Numerical methods: Finite difference beam propagation

- In finite difference beam propagation, the purpose here is to model the propagation of light from a known source in a defined refractive index structure.

- The model uses the same finite difference approach to calculate the field propagation knowing the input source.

- The wave propagation equation is

\[
\nabla^2 E(x, z) + \omega \mu_0 \epsilon_0 n^2(x) E(x) = 0
\]

\[
\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial z^2} + k_0^2 n(x, z)^2 E = 0
\]

- Applying the finite difference approximation we obtain the following

\[
\frac{E_{m+1,l} - 2E_{m,l} + E_{m-1,l}}{\Delta_x^2} + \frac{E_{m,l+1} - 2E_{m,l} + E_{m,l-1}}{\Delta_z^2} + k_0^2 n_{m,l} E_{m,l} = 0
\]

- Here index \(m\) refers to the \(x\) axis and \(l\) refers to the \(z\) axis

The field is known at \(z=0\)

Using the input field the wave-propagation equation with finite difference approximation to calculate the field in the working space

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Numerical methods: Finite difference beam propagation

- Re-arranging the above equation, we obtain

\[ E_{m,l+1} = -(E_{m+1,l} + E_{m-1,l}) \frac{\Delta z}{\Delta x} - E_{m,l-1} + \left( \frac{\Delta z}{\Delta x} + 1 - \Delta z^2 k_o n_m^2 \right) E_{m,l} \]

- Hence, to calculate the field in the next z step, one needs to know the field distribution in the previous two steps (l and l-1).

- That also indicates that the input field has to be defined at two locations \( z=0 \) and \( z=-\Delta z \), where \( \Delta z \) is the sampling step in the z direction.

- From the input fields, the next field at location \( \Delta z \) is calculated (or index l=1).

- From the field at \( z=0 \) and \( z=\Delta z \), the new field at location \( z=2\Delta z \) is calculated.

- The process repeats till the field is calculated across the desired space.
Numerical methods: FD-BPM Boundary conditions

- The highlighted two terms in the equation above would cause errors at the top and bottom boundaries of the calculation space (m=0 and m=M), where M is the total number of points in the x direction.

- The requires assigning values for the field outside the boundary (for m=-1 and m=M+1).

- If the field profile is chosen such that the amplitude vanishes at the boundaries, then the boundary condition can be set to have the fields equal zero.

- \[ E_{m,l+1} = (E_{m+1,l} + E_{m-1,l}) \frac{\Delta z^2}{\Delta x} - E_{m,l-1} + \left( \frac{\Delta z^2}{\Delta x} + 1 - \Delta z^2 k_o^2 n_m^2 \right) E_{m,l} \]

- \[ E_{-1,l} = 0 \quad \text{and} \quad E_{M+1,l} = 0 \]
FD-BPM: Gaussian beam propagation

- Defining the input, one needs to define the profile, wavelength and location.

- If a Gaussian beam is assumed to be located at the input, at a position $x_0$ and with a beam width of $w_0$, the field at both input locations can be assumed to be

$$E_{input}(x) = \exp\left(-\frac{(x-x_0)^2}{2w_0^2}\right)$$

- The calculations on the side shows the solution for an input Gaussian of width 4 $\mu$m, wavelength of 1 $\mu$m, and medium index of 1.45.

- The calculations space is 20 $\mu$m x 50 $\mu$m.
FD-BPM: Slow varying amplitude assumption

- The calculations in this example showed an oscillating Gaussian beam with period of $\lambda/n$.
- As the oscillations is varying fast, that puts a requirements of having a small step in $z$ ($\Delta z$).
- Hence, the calculations time would be large for tracing large structure.
- One way to solve this issue is to assume a solution of the electric field in the form of

$$E(x, z) = A(x, z)\exp(-ik_o n_o z)$$

- In this representation $A(x,z)$ is a slow varying amplitude and $n_o$ is an average refractive index.
- Substituting this form in the wave equation

$$\frac{\partial^2 A}{\partial x^2} \exp(-ik_o n_o z) + \frac{\partial^2 A}{\partial z^2} A \exp(-ik_o n_o z) + k_o^2 n(x, z)^2 A \exp(-ik_o n_o z) = 0$$

$$\frac{\partial^2 A}{\partial x^2} \exp(-ik_o n_o z) + \frac{\partial^2 A}{\partial z^2} \exp(-ik_o n_o z) - 2ik_o n_o \frac{\partial A}{\partial z} \exp(-ik_o n_o z) + k_o^2 (n(x, z)^2 - n_o^2) A \exp(-ik_o n_o z) = 0$$

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2} - 2ik_o n_o \frac{\partial A}{\partial z} + k_o^2 (n(x, z)^2 - n_o^2) A = 0$$
**FD-BPM: Slow varying amplitude assumption**

- Applying the slowly varying amplitude assumption that the second derivative in z of A is much smaller than the first derivative term, the following expression is obtained

\[
\frac{\partial^2 A}{\partial x^2} - 2 i k_o n_o \frac{\partial A}{\partial z} + k_o^2 \left( n(x, z)^2 - n_o^2 \right) A = 0
\]

- Using finite difference approximation

\[
\frac{A_{m+1,l} - 2 A_{m,l} + A_{m-1,l}}{\Delta_x^2} - 2 i k_o n_o \frac{A_{m,l+1} - A_{m,l-1}}{2 \Delta_z} + k_o^2 \left( n_{m,l}^2 - n_o^2 \right) A_{m,l} = 0
\]

- Re-arranging these terms

\[
A_{m,l+1} = \frac{\Delta_z}{i k_o n_o} \left[ \frac{(A_{m+1,l} + A_{m-1,l})}{\Delta_x^2} + i k_o n_o \frac{A_{m,l-1}}{\Delta_z} + \left( \frac{2}{\Delta_x^2} + k_o^2 \left( n_{m,l}^2 - n_o^2 \right) \right) A_{m,l} \right]
\]

- In this representation A is a slowly varying amplitude.

- Hence, larger values of \( \Delta z \) can be used.

- That reduces dramatically the computation effort.
**FD-BPM:** Slow varying amplitude assumption

- Using this method to solve for the propagation of the same Gaussian beam as before, the graph below shows the amplitude solution.

- The step size used here is 10 times larger than before.

- The computation effort is reduced by a factor of 10.