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Fourier Series

*Photo from *http:www.uh.eduenginesepi186.htm*

Periodic Functions

Definition (Periodic Function)

A function $f(x)$ is periodic function with period p if

$$f(x+p) = f(x), \qquad \text{for all } x. \tag{1}$$

Note that the definition in (1) implies that $f(x)$ has also period $2p$ since,

$$f(x+2p) = f(x+p+p) = f((x+p)+p) = f(x+p) = f(x).$$

The two last equalities come from (1). In fact, (1) implies that

$$f(x+np) = f(x),$$

for $n = \pm 1, \pm 2, \ldots$.

Example FS-1: Given $f(x)$ in the figure below, what is the period of $f(x)$

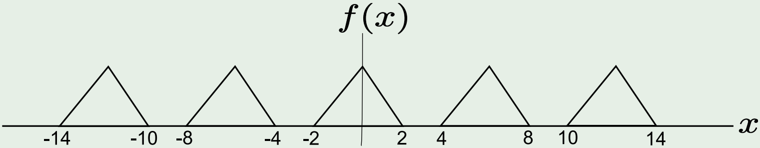


Figure 1: Triangular function $f(x)$ in Example FS-1

$$
\begin{aligned}
f(x=0) &= f(x=6) = f(x=0+6) \\
&= f(x=12) = f(x=0+2\cdot 6) \\
&= f(x=-12) = f(x=0+-2\cdot 6)
\end{aligned}
$$

Hence, the period of $f(x)$ is 6.

Periodic Function

Example FS-2: sin(x) and cos(x)

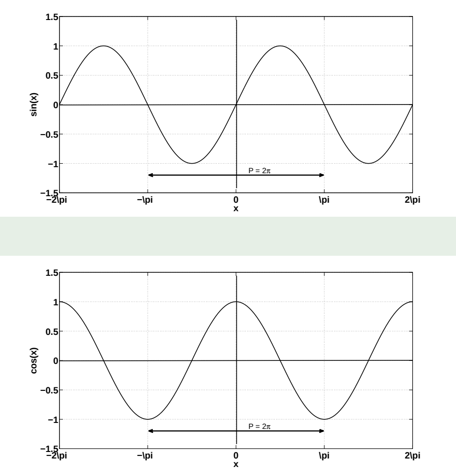


Figure 2: Sine and cosine function for $-\pi \leq x \leq \pi$

Periodic Function

Example FS-3: Non-periodic functions

$$f(x) = x$$
$$f(x) = x^2$$
$$f(x) = x^3$$
$$f(x) = \log(x)$$

Fourier Series

Fourier series of $f(x)$ is the representation of $f(x)$ in term of the summation of basic functions which are trigonometric functions.

Definition (Fourier Series)

Given function $f(x)$. Fourier seris of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad L = P/2 \tag{2}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, \mathrm{d}x \tag{3}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x)\, \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad n = 1, 2, \ldots \tag{4}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x)\, \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad n = 1, 2, \ldots \tag{5}$$

Example FS-3: Find Fourier series of the following function $f(x)$

$$f(x) = \begin{cases} -k & , \pi < x < 0 \\ k & , 0 < x < \pi. \end{cases} \quad \text{and } f(x+2\pi) = f(x).$$

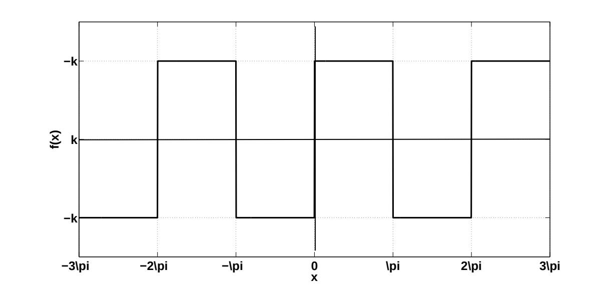


Figure 3: Square function $f(x)$ in Example FS-3

- $f(x)$ is periodic function with period $P = 2\pi \Rightarrow L = \pi$.

Fourier Series

- Compute a_0.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$
$$= \frac{1}{2(\pi)}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2(\pi)}\left[\int_{-\pi}^{0}(-k)dx + \int_{0}^{\pi}(k)dx\right]$$
$$= \frac{1}{2\pi}[-k(0-(-\pi)+k(\pi-0)]$$
$$= \frac{1}{2\pi}[-k\pi+k\pi] = 0$$

- Compute a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\frac{n\pi x}{\pi})dx$$

Fourier Series

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos(nx) + \int_{0}^{\pi}(k)\cos(nx)\right] \\
&= \frac{1}{\pi}\left[(-k)\left(\frac{\sin(nx)}{n}\right)\Bigg|_{-\pi}^{0} + (k)\left(\frac{\sin(nx)}{n}\right)\Bigg|_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{-k}{n}\left(\overbrace{\sin(0)}^{0} - \overbrace{\sin(-n\pi)}^{0}\right) + \frac{k}{n}\left(\overbrace{\sin(n\pi)}^{0} - \overbrace{\sin(0)}^{0}\right)\right] \\
&= \frac{1}{\pi}(0+0) = 0 \quad \blacksquare
\end{aligned}
$$

Fourier Series

- Compute b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(\frac{n\pi x}{\pi})dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin(nx) + \int_{0}^{\pi}(k)\sin(nx)\right]$$

$$= \frac{1}{\pi}\left[(-k)\left(\frac{-\cos(nx)}{n}\right)\Big|_{-\pi}^{0} + (k)\left(\frac{-\cos(nx)}{n}\right)\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-k}{n}\left(-\overbrace{\cos(0)}^{1} + \cos(-n\pi)\right) + \frac{k}{n}\left(-\cos(n\pi) + \overbrace{\cos(0)}^{1}\right)\right]$$

Recall $\cos(\theta) = \cos(-\theta)$,

$$= \frac{2k}{n\pi}(1 - \cos(n\pi)) \quad \blacksquare$$

- Fourier series of $f(x)$ is

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \\
&= \sum_{n=1}^{\infty} \overbrace{\frac{2k}{n\pi}(1-\cos(n\pi))}^{b_n} \sin(nx) \\
&= \frac{4k}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right) \quad \blacksquare \qquad (6)
\end{aligned}
$$

Note that for this example, when n is even number, b_n is equal to 0. Hence, its corresponding term vanishes from the summation.

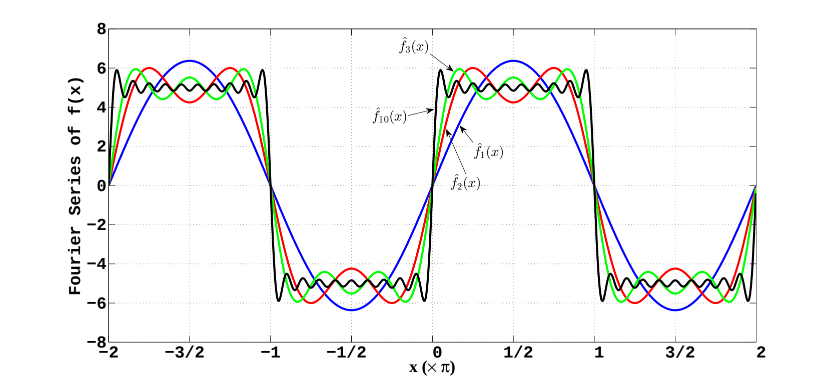


Figure 4: Fourier series of $f(x)$ in Example FS-3 results from (6) (given $k = 6$)

Note also that $\hat{f}_n(x)$ in the above figure are the fourier series of $f(x)$ when the first n terms are used on (6).

Example FS-4: Find Fourier series of the $u(x)$

$$u(x) = \begin{cases} 0 & , -L < x < 0 \\ E\sin(\omega x) & , 0 < x < L. \end{cases}$$

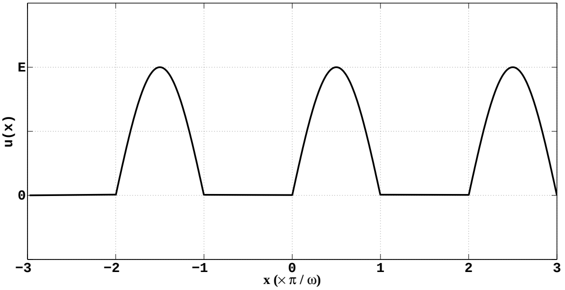


Figure 5: Function $u(x)$ in Example FS-4 (given $L = 1$)

- $u(x)$ is periodic function with period $P = 2L = \frac{2\pi}{\omega} \Rightarrow L = \frac{\pi}{\omega}$.
- Compute a_0.

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2(\pi/\omega)}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(x)dx = \frac{\omega}{2\pi}\left[\int_0^{\frac{\pi}{\omega}} E\sin(\omega x)dx\right] \\ &= \frac{\omega}{2\pi}\left[-\frac{E\sin(\omega x)}{\omega}\right]\Bigg|_0^{\frac{\pi}{\omega}} = -\frac{E}{2\pi}(\cos(\pi)-\cos(0)) \\ &= \frac{E}{\pi} \quad \blacksquare \end{aligned}$$

- Compute a_n

$$a_n = \frac{1}{\pi/\omega}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(x)\cos\left(\frac{n\pi x}{\pi/\omega}\right)dx$$

Fourier Series

$$
\begin{aligned}
a_n &= \frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}} E\sin(\omega x)\cos(n\omega x)dx \\
&= \frac{\omega E}{\pi}\int_0^{\frac{\pi}{\omega}} \left[\frac{\sin[(1+n)\omega x]}{2} + \frac{[(1-n)\omega x]}{2}\right]dx \\
&= \frac{\omega E}{2\pi}\left(\frac{-\cos[(1+n)\omega x]}{(1+n)\omega}\Bigg|_0^{\frac{\pi}{\omega}} + \frac{-\cos[(1-n)\omega x]}{(1-n)\omega}\Bigg|_0^{\frac{\pi}{\omega}}\right) \\
&= \frac{\omega E}{2\pi}\left[\frac{1}{(1+n)\omega}(-\cos[(1+n)\pi]+\cos(0))\right. \\
&\qquad\qquad \left. + \frac{1}{(1-n)\omega}(-\cos[(1-n)\pi]+\cos(0))\right] \\
&= \frac{\omega E}{2\pi}\left[\frac{1-\cos[(1+n)\pi]}{1+n} + \frac{1-\cos[(1-n)\pi]}{1-n}\right]
\end{aligned}
$$

Hence,

$$a_n = \begin{cases} 0, & n \text{ odd} \\ \frac{2E}{\pi(1+n)(1-n)}, & n \text{ even.} \end{cases} \qquad \blacksquare$$

- Compute b_n

$$\begin{aligned} b_n &= \frac{1}{\pi/\omega} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(x) \sin\left(\frac{n\pi x}{\pi/\omega}\right) dx \\ &= \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} E \sin(\omega x) \sin(n\omega x) dx \\ &= \frac{\omega E}{\pi} \int_0^{\frac{\pi}{\omega}} \left[\frac{\cos[(1-n)\omega x]}{2} - \frac{[(1+n)\omega x]}{2}\right] dx \end{aligned}$$

$$b_n = \frac{\omega E}{2\pi}\left(\frac{\sin[(1-n)\omega x]}{(1-n)\omega}\bigg|_0^{\frac{\pi}{\omega}} - \frac{\sin[(1+n)\omega x]}{(1+n)\omega}\bigg|_0^{\frac{\pi}{\omega}} \right)$$

$$= \frac{\omega E}{2\pi}\left[\frac{1}{(1-n)\omega}\left(\sin[(1-n)\pi] + \sin(0)\right) - \frac{1}{(1+n)\omega}\left(\sin[(1+n)\pi] - \sin(0)\right) \right]$$

$$= \frac{\omega E}{2\pi}\left[\frac{\sin[(1-n)\pi]}{1-n} - \frac{\sin[(1+n)\pi]}{1+n} \right]$$

Notice that $b_n = 0$ for all n, except for $n = 1$. When $n = 1$, use L'Hospitals' rule, we get

$$b_1 = \frac{\omega E}{2\pi}\left(\frac{\cos[(1-n)\pi](-\pi)}{-1} \right) = \frac{E}{2} \qquad \blacksquare$$

- Fourier series of $f(x)$ is

$$u(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \quad (7)$$

$$= \underbrace{\frac{E}{\pi}}_{a_0} + \underbrace{\frac{E}{2}}_{b_1} \sin(\omega x) + \sum_{n=1}^{\infty} \underbrace{\frac{2E}{\pi(1-n^2)}}_{a_n} \cos(n\omega x) \quad (8)$$

$$= \frac{E}{\pi} + \frac{E}{2}\sin(\omega x) - \frac{2E}{3\pi}\cos(2\omega x) - \frac{2E}{15\pi}\cos(3\omega x) - \ldots \quad \blacksquare \quad (9)$$

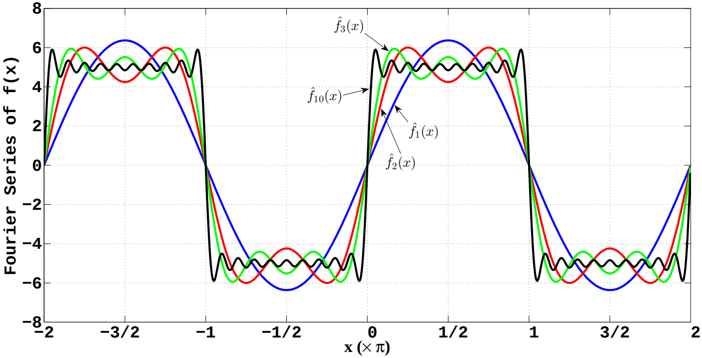


Figure 6: Fourier series of $u(x)$ in Example FS-4 (given $L = 1$)

Note that $\hat{u}_n(x)$ in the above figure are the fourier series of $u(x)$ when the first n terms are used on (9).

Theorem (Sum and Scalar Multiple)

- *The Fourier coefficients of a sum of $f_1(x)$ and $f_2(x)$ are the sums of the corresponding Fourier coeffcients of $f_1(x)$ and $f_2(x)$*
- *The Fourier coefficients of $cf_(x)$ are c times the corresponding Fourier coeffcients of $f(x)$.*

In conclusion, given that

$$a_0^{(1)}, \{a_n^{(1)}, b_n^{(1)}\}_{n=1}^{\infty} \text{ are Fourier coefficients of } f_1(x)$$
$$a_0^{(2)}, \{a_n^{(2)}, b_n^{(2)}\}_{n=1}^{\infty} \text{ are Fourier coefficients of } f_2(x).$$

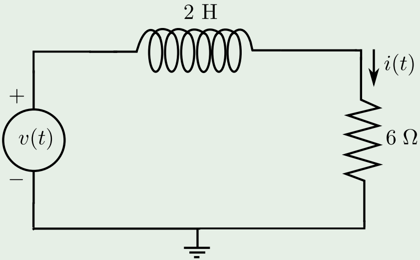
If $f_3(x) = pf_1(x) + qf_2(x)$, where p and q are scalar, then from the theorem above, the Fourier coefficients of $f_3(x)$ are as follow.

$$a_0^{(3)} = pa_0^{(1)} + qa_0^{(2)}$$
$$a_n^{(3)} = pa_n^{(1)} + qa_n^{(2)}$$
$$b_n^{(3)} = pb_n^{(1)} + qb_n^{(2)}$$

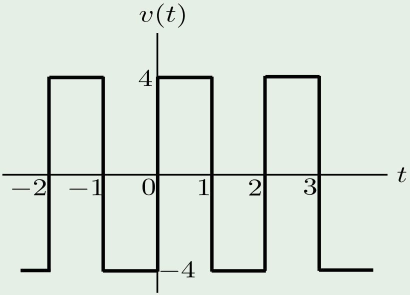
Application of Fourier Series: Force Response of Periodic Function

Force response of periodic function in RL-circuit

Given RL circuit in the Figure (a) and the source $v(t)$ is the periodic signal as shown in Fig. (b). Find the forced response $i(t)$.



(a)



(b)

Figure 7: (a) Given RL circuit (b) source waveform $v(t)$

Application of Fourier Series: Force Response of Periodic Function

As we already do the similar example earlier, the Fourier representation of $v(t)$ is as follow.

$$v(t) = \frac{16}{\pi}\left(\sin(\pi t) + \frac{\sin(3\pi t)}{3} + \frac{\sin(5\pi t)}{5} + \dots\right)$$
$$= \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi t]}{2n-1} = \sum_{n=1}^{\infty} v_n(t) \qquad (10)$$

We can see that each Fourier component of $v(t)$ is the sinusoidal signal. Hence, we can apply the idea of Phasor [2] to find the force response $i(t)$. From (10), the $(2n-1)^{th}$ harmonic of $v(t)$ is

$$v_n(t) = \frac{16}{\pi}\left(\frac{\sin[(2n-1)\pi t]}{2n-1}\right),$$

with frequency $\omega_n = (2n-1)\pi$ rad/s.

Application of Fourier Series: Force Response of Periodic Function

Then, the phasor voltage of $v_n(t)$ is

$$\mathbf{V}_n = \frac{16}{(2n-1)\pi}\angle 0^\circ$$

and the total impedance in the circuit is

$$\begin{aligned}\mathbf{Z}(j\omega_n) &= 6 + j\omega_n L = 6 + j((2n-1)\pi)2 \\ &= 2\sqrt{9+\pi^2(2n-1)^2}\angle \tan^{-1}\left(\frac{(2n-1)\pi}{3}\right)\end{aligned}$$

Thus, the phasor of $(2n-1)^{th}$ harmonic of $i(t)$ can be computed by

$$\begin{aligned}\mathbf{I}_n &= \frac{\mathbf{V}_n}{\mathbf{Z}(j\omega_n)} \\ &= \frac{8}{(2n-1)\pi\sqrt{9+\pi^2(2n-1)^2}}\angle -\tan^{-1}\left(\frac{(2n-1)\pi}{3}\right)\end{aligned}$$

Application of Fourier Series: Force Response of Periodic Function

In time-domain, $(2n-1)^{th}$ harmonic of $i(t)$ is

$$i_n(t) = |\mathbf{I}_n| \sin(\omega_n t - \phi_n),$$

where $|\mathbf{I}_n| = \frac{8}{(2n-1)\pi\sqrt{9+\pi^2(2n-1)^2}}$ and $\phi_n = \tan^{-1}\left(\frac{(2n-1)\pi}{3}\right)$.

Finally, by superposition principle, the force response $i(t)$ can be computed by

$$i(t) = \sum_{n=1}^{\infty} i_n(t) = \sum_{n=1}^{\infty} |\mathbf{I}_n| \sin(\omega_n t - \phi_n) \qquad \blacksquare$$

Application of Fourier Series: Average Power of Periodic Signal

Average Power of Periodic Signal

Refer to the RL circuit in previous example, we already know the input voltage $v(t)$ and the terminal current $i(t)$ are

$$v(t) = \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi t]}{2n-1} \text{ V.} \tag{11}$$

$$i(t) = \sum_{n=1}^{\infty}|\mathbf{I}_n|\sin(\omega_n t - \phi_n) \text{ A,} \tag{12}$$

where $|\mathbf{I}_n| = \frac{8}{(2n-1)\pi\sqrt{9+\pi^2(2n-1)^2}}$ and $\phi_n = \tan^{-1}\left[\frac{(2n-1)\pi}{3}\right]$. To compute average power the source delivers to the circuit, we have to deal with the product of two Fourier series. This may be complicated.

Application of Fourier Series: Average Power of Periodic Signal

$$P_{ave} = \frac{1}{T}\int_0^T v(t)i(t)dt \quad = \frac{16}{\pi T}\int_0^T \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mathbf{I}_m \left|\frac{\sin[(2n-1)\pi t]}{2n-1}\right| \sin(\omega_m t - \phi_m)dt$$

However, it is simple than that. Suppose that we have two Fourier series in the following form.

$$f = A_{dc} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

$$g = B_{dc} + \sum_{m=1}^{\infty} B_m \cos(m\omega_0 t + \theta_m)$$

Then, using the properties of the integrals of sinusoidal functions [2], it can be shown that

$$\frac{1}{T}\int_0^T fg\, dt = A_{dc}B_{dc} + \sum_{n=1}^{\infty} \frac{A_n B_n}{2} \cos(\phi_n - \theta_n).$$

Application of Fourier Series: Average Power of Periodic Signal

Thus, by letting $f = v$, $g = i$, $A_{dc} = V_{dc}$, $A_n = V_n$, $B_{dc} = I_{dc}$, and $B_n = I_n$, the average power is given by

$$P_{ave} = \frac{1}{T}\int_0^T v(t)i(t)dt = V_{dc}I_{dc} + \sum_{n=1}^{\infty} \frac{V_n I_n}{2}\cos(\phi_n - \theta_n). \tag{13}$$

Note that the above formula is valid only when v and i are function of cosine. Now, we are ready to compute average power of RL circuits in previous example. Sine terms of $v(t)$ and $i(t)$ in (12) may be converted to cosine terms by subtracting $\pi/2$ from their phase angles. Since by using (13), both $v(t)$ and $i(t)$ need to do the same operation, $\pi/2$ will be canceled out. Hence, substituting (12) into (13), the average power is simply computed as

Application of Fourier Series: Average Power of Periodic Signal

$$P_{ave} = \sum_{n=1}^{\infty} \frac{16}{2(2n-1)\pi} \left[\frac{8}{(2n-1)\pi\sqrt{9+\pi^2(2n-1)^2}} \right] \cos\left[\tan^{-1} \frac{(2n-1)\pi}{3} \right]$$

$$= \frac{192}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2[9+\pi^2(2n-1)^2]} \text{ W} \qquad \blacksquare.$$

Summary

- Fourier series is commonly used in analysis of periodic functions or signals.
- Fourier series is computed from (2)-(5).
- Generally speaking, Fourier series is another way to represent periodic function in term of the summation of sine and cosine functions at various harmonics frequencies of that function.
- The more terms used in Fourier series, the more closer of series to the original function.
- To be able to compute Fourier series, good knowledge of integral of sinusoidal function is required.
- Fourier series will be really helpful in various applications, more specifically the applications that related to periodic phenomena involving differential equations. For example, RLC circuit with periodic excitation and mechanical vibration problems.

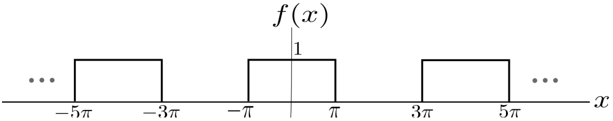
In Exercise 1-4, given periodic function $f(x)$, do the following 1) sketch $f(x)$, 2) find period of $f(x)$ and, 3) calculate its Fourier series ($a_0, a_n,$ and b_n and also Eq.(2))

Exer 1. $f(x) = x^2 \quad \text{for} \quad -1 \leq x \leq 1,$

$$f(x+2) = f(x)$$

Exer 2. $f(x) = \begin{cases} 0 & , -2 < x < 0 \\ x & , 0 < x < 2, \end{cases}$ and $f(x+4) = f(x)$.

Exer 3. $f(x)$ is given as the following figure.



Exer 4. $f(x) = \begin{cases} -x & , -\pi < x < 0 \\ x & , 0 < x < \pi, \end{cases}$ and $f(x+2\pi) = f(x)$.

Exer 5. Given $f(x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$,

$$f(x+\pi) = f(x).$$

Find period of $f(x)$ and calculate Fourier series of $f(x)$.

Exer 6. Use result in Exer 5. to plot Fourier series using graphical software tool, i.e., Microsoft Excel. Do the following 3 cases.

(a) Use $n = 1$
(b) Use $n = 1, 2,$ and 3
(c) Use $n = 1, 2, 3, 4,$ and 5

Observe that when n increases, how Fourier series gets closer to function $f(x)$.