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Fourier Integral and Transform

*Photo from WikiPedia

- Fourier series is an important tool for solving problems relating with *periodic* functions/signals.
- However, many mathematical problems involve with *non-periodic* functions/signals.
- The idea of Fourier series can be extended to be used for non-periodic functions/signals. Then, it is called **Fourier Integral**. This can be demonstrated from the following example.

Example FI-1: $f_L(x)$ is Rectangular wave with period L.

$$f_L(x) = \begin{cases} 0, & -L < x < -1 \\ 1, & -1 < x < 1 \\ 0, & 1 < x < L. \end{cases}$$

$\Rightarrow f_L(x)$ has a period $(P) = 2L$

Fourier Integral

We already know that

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} \cos(n\pi x/L)dx = \frac{2}{L}\frac{\sin(n\pi/L)}{n\pi/L}$$

$$b_n = 0 \text{ (because } f_L(x) \text{ is even function)}$$

The sequence of Fourier coefficients, which in this case is a_n, is called **amplitude spectrum**. If we plot graph of a_n by setting x-axis as $\omega_n = n\pi/L$ and y-axis as the value of a_n, for $n = 1, 2, \ldots$. Then, we get the plot as figure below.

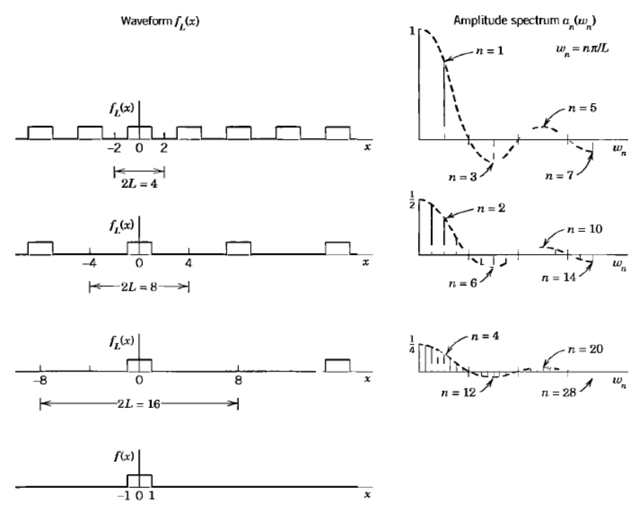


*Figure from *Advanced engineering mathematics* by Erwin Kreyszig.

Figure 1: Rectangular waveform for various values of L in Example FI-1

Fourier Integral

As you can see, when L increases, amplitude spectrum is getting more closer. If $L \to \infty$, amplitude spectrum becomes continuous function and also $f_L(x)$ becomes non-periodic function. Hence, we already have Fourier series of non-periodic function by taking $L \to \infty$. Fourier series of non-periodic function is called **"Fourier Integral"**.

Definition (Fourier Integral)

Given non-periodic function $f(x)$, its Fourier integral is obtained from the following formula

$$f(x) = \int_0^{\infty} [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\, d\omega \tag{14}$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv \tag{15}$$

Fourier Integral

*** See the how the Fourier integral formula is derived from Fourier series when $L \to \infty$ in [1].

Example: FI-2 Find Fourier integral of the following rectangular function $f(x)$

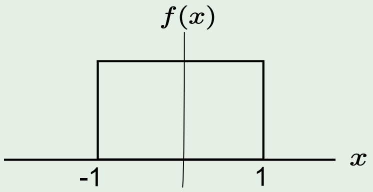


Figure 2: Rectangular function considering in Example FI-2.

$$f_L(x) = \begin{cases} 1, & \|x\| < 1 \\ 0, & \|x\| > 1. \end{cases}$$

Fourier Integral

$$
\begin{aligned}
A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv = \frac{1}{\pi}\int_{-1}^{1}(1)\cos(\omega v)dv \\
&= \left.\frac{\sin(\omega v)}{\omega\pi}\right|_{-1}^{1} = \frac{1}{\omega\pi}\left(\sin(\omega) - \sin(-\omega)\right) \\
&= \frac{2\sin(\omega)}{\omega\pi}.
\end{aligned}
$$

$$
\begin{aligned}
B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv = \frac{1}{\pi}\int_{-1}^{1}(1)\sin(\omega v)dv \\
&= -\frac{1}{\pi}\left.\frac{\cos(\omega v)}{\omega}\right|_{-1}^{1} = -\frac{1}{\omega\pi}\left(\cos(\omega) - \cos(-\omega)\right) \\
&= 0.
\end{aligned}
$$

Fourier Integral

Hence, Fourier integral of $f(x)$ is

$$f(x) = \int_0^\infty \frac{2\sin(\omega)}{\omega\pi} \cdot \cos(\omega x) d\omega \quad \blacksquare$$

Similar to Fourier series which we can consider only some partial terms of the summation, for Fourier integral, if we want to consider only some parts of the Fourier integral, we replace ∞ in the upper limit with some constant a and denote the result of the integration by $\tilde{f}(x)$, i.e.,

$$f(x) \approx \tilde{f}(x) = \int_0^a \frac{2\sin(\omega)}{\omega\pi} \cdot \cos(\omega x) d\omega$$

The results of the Fourier integral for various value of a are shown in the following figure.

Fourier Integral

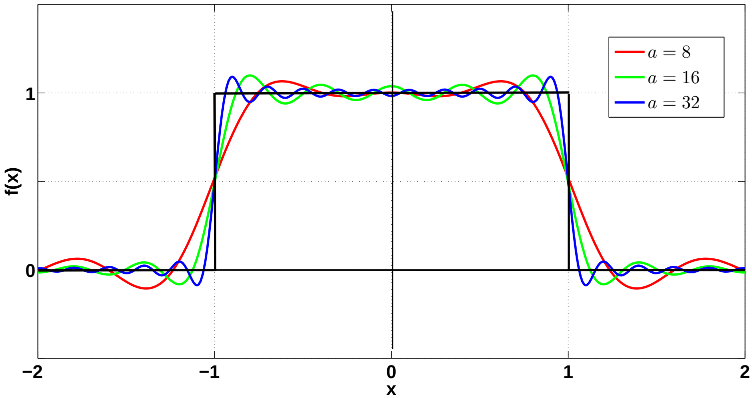


Figure 3: $\hat{f}(x)$ for various values of a.

Ask yourself !!!

- What difference between Fourier series and Fourier integral?

In this section, we will introduce another one of the most important transformation generally used in the engineering problems, called **Fourier Transform**. The Fourier transform can be derived from Fourier integral as follow.

Derviation of Fourier transform

Consider Fourier integral of $f(x)$

$$f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\, d\omega \tag{16}$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv \tag{17}$$

Substitute $A(\omega)$ and $B(\omega)$ into (16).

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(v)\underbrace{[\cos(\omega v)\cos(\omega x) + \sin(\omega v)\sin(\omega x)]}_{\cos(\omega v - \omega x)} dv d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \underbrace{\left[\int_{-\infty}^\infty f(v)\cos(\omega v - \omega x)dv\right]}_{\text{let's call } F(\omega)} d\omega \tag{18}$$

Notice that $F(\omega)$ in (18) is *even function* so that $\int_0^\infty F(\omega)d\omega = \frac{1}{2}\int_{-\infty}^\infty F(\omega)d\omega$. Hence, from (18).

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v)\cos(\omega x - \omega v)dv\right] d\omega \tag{19}$$

Also, observe that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(\omega x-\omega v)dv\right]d\omega = 0, \tag{20}$$

since the value in $[\ldots]$ is *odd function*. Thus, $\int_{-\infty}^{\infty}[\ldots]d\omega = 0$. Recall that

$$e^{ix} = \cos(x) + i\sin(x), \quad i = \sqrt{-1}. \tag{21}$$

Now, from (20), without any effects we can add (20) to (19) as follow.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(\omega x-\omega v)dv\right]d\omega + \frac{i}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(\omega x-\omega v)dv\right]d\omega \tag{22}$$

Using (21), (22) reduces to

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\nu)e^{i\omega(x-\nu)}\,d\nu\,d\omega$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\nu)e^{-i\omega\nu}\,d\nu\right]}_{\hat{f}(\omega)} e^{i\omega x}\,d\omega, \quad (23)$$

where $\hat{f}(\omega)$ is defined as Fourier transform of $f(x)$.

Definition (Fourier Transfrom and Inverse Fourier Transform)

Given function $f(x)$, the **Fourier transform** of $f(x)$, denoted by $\hat{f}(x)$ is defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{24}$$

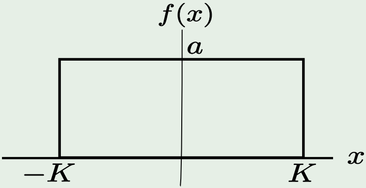
Also, given $\hat{f}(x)$, function $f(x)$ can be obtained back by taking **inverse Fourier transform** of $\hat{f}(x)$ defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{i\omega x} d\omega \tag{25}$$

Notation: $$f(x) \overset{\mathscr{F}}{\Longleftrightarrow} \hat{f}(\omega)$$

$$\hat{f}(\omega) = \mathscr{F}\{f(x)\} \text{ and } f(x) = \mathscr{F}^{-1}\{\hat{f}(\omega)\}$$

Example FT-1: Fourier transform of rectangular function



$$f(x) = \begin{cases} a, & \|x\| < K \\ 0, & \text{otherwise.} \end{cases}$$

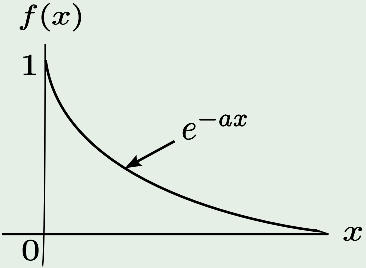
Figure 4: $f(x)$ for Example FT-1

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-K}^{K} (a) e^{-i\omega x} dx$$

$$= \frac{a}{\sqrt{2\pi}} \left(\frac{e^{-i\omega x}}{-i\omega} \right) \Big|_{-K}^{K} = \frac{a}{\sqrt{2\pi}} \left(\frac{e^{i\omega K} - e^{-i\omega K}}{i\omega} \right)$$

$$= a\sqrt{\frac{2}{\pi}} \frac{\sin(\omega K)}{\omega} \quad \blacksquare$$

Example FT-2: Fourier transform of exponential function



$$f(x) = \begin{cases} e^{-ax}, & x > 0 \\ 0, & \text{otherwise}, \end{cases}$$

where $a > 0$.

Figure 5: $f(x)$ for Example FT-2

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \cdot e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(a+i\omega)x} dx$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right) \Bigg|_0^{\infty} = \frac{1}{\sqrt{2\pi}} \left(\frac{(e^{-\infty} - 1}{-(a+i\omega)} \right)$$

$$= \frac{1}{\sqrt{2\pi}(a+i\omega)} \qquad \blacksquare$$

$f(x),\ x \geq 0$	$\hat{f}(\omega)$
$\begin{cases} 1, & \lVert x \rVert < K \\ 0, & \text{otherwise.} \end{cases}$	$\frac{2}{\pi}\frac{\sin(b\omega)}{\omega}$
$\begin{cases} 1, & b < x < c \\ 0, & \text{otherwise.} \end{cases}$	$\frac{e^{-ib\omega} - e^{-c\omega}}{i\omega\sqrt{2\pi}}$
$\begin{cases} e^{-ax}, & x > 0, a > 0 \\ 0, & \text{otherwise.} \end{cases}$	$\frac{1}{\sqrt{2\pi(a+i\omega)}}$
$\begin{cases} e^{ax}, & b < x < c \\ 0, & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\omega)c} - e^{(a-i\omega)b}}{\sqrt{2\pi(a+i\omega)}}$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-\omega^2/4a}$

$(x),\ x \geq 0$	$\hat{f}(\omega)$
$e^{-a\lvert x\rvert}, a > 0$	$\frac{2}{\pi}\frac{a}{\omega^2 - a^2}$
$\frac{1}{1+a^s+x^2}$	$\sqrt{\frac{\pi}{2}}e^{-\lvert\omega/a\rvert}$
$xe^{-ax}, a > 0, x > 0$	$\frac{1}{\sqrt{2\pi}(a+\omega i)^2}$
$xe^{ax}, a > 0, x < 0$	$\frac{-1}{\sqrt{2\pi}(a-\omega i)^2}$
$\frac{\sin(ax)}{ax}$	$\begin{cases} a\sqrt{\frac{\pi}{2}}, & \lvert\omega\rvert < a \\ 0, & \text{otherwise.} \end{cases}$

Table-1: The Fourier transform $\hat{f}(\omega)$ of some common functions $f(x)$

Physical Interpretation of Fourier Transform: Spectrum

- $\hat{f}(\omega)$ is called **"Spectral Density"**. Think of the superposition of all possible frequencies of sinusoidal functions, $\hat{f}(\omega)$ tells us the intensity of frequency interval between ω and $\omega + \Delta\omega$, where $\Delta\omega$ is very small and fixed, that $f(x)$ is composed of. In other word, $\hat{f}(\omega)$ tells us that in order to generate function or signal $f(x)$ by summing of all possible frequencies of sinusoidal functions, how large (intensity) of the sinusoidal functions at frequency interval between ω and $\omega + \Delta\omega$ should be.
- $\hat{f}(\omega)$ is also called **"Amplitude Spectrum"** similar to what Fourier series coefficients are called. However, as we will see, the amplitude spectrum obtained from Fourier series coefficients are *discrete* where only the frequencies multiple of π/L are exist while amplitude spectrum obtained from Fourier transform is *continuous* where all frequencies are possible to exist.

Fourier Transform

- Since usually $\hat{f}(\omega)$ is complex-valued, it might be more convenient to use the magnitude and phase of $\hat{f}(\omega)$ instead.

$$\hat{f}(\omega) = |\hat{f}(\omega)|e^{-i\Theta(\omega)},$$

where $|\hat{f}(\omega)|$ and $\Theta(\omega)$ are called **Magnitude Spectrum** and **Phase Spectrum**, respectively.

Example FT-3: Compute magnitude spectrum of the following function

$$f(x) = \begin{cases} e^{-ax}, & x > 0 \\ 0, & \text{otherwise}, \end{cases}$$

From previous example, we already know that

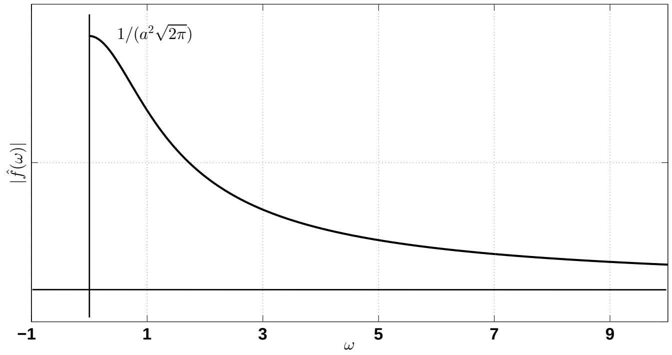
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}(a + i\omega)}.$$

Then, magnitude spectrum can be found as follow.

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}(a+i\omega)} \times \frac{a-i\omega}{a-i\omega} = \frac{a-i\omega}{\sqrt{2\pi}\,(a^2+\omega^2)} \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{a^2+\omega^2} + i\frac{-\omega}{a^2+\omega^2}\right] \end{aligned}$$

Hence,

$$|\hat{f}(\omega)| = \frac{1}{\sqrt{2\pi}(a^2+\omega^2)}\sqrt{a^2+\omega^2} = \frac{1}{\sqrt{2\pi}(a^2+\omega^2)}$$



Some Properties of Fourier Transform

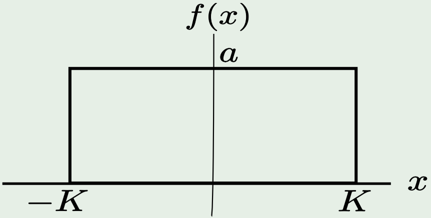
Theorem (Linearity of Fourier Transform)

For any functions $f(x)$ and $g(x)$ whose Fourier transform exist and any constant a and b, it can be shown that

$$\mathscr{F}\{af(x)+bg(x)\} = a\mathscr{F}\{f(x)\}+b\mathscr{F}\{g(x)\}$$

Example FT-4: Fourier transform of the sum of rectangular functions

Denote the rectangular function as follow.



$$\equiv \quad a\cdot\text{rect}\left(\frac{x}{K}\right)$$

Figure 7: Fourier transform pair of rectangular function

Some Properties of Fourier Transform

From previous example, we know that

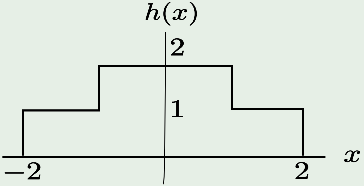
$$\mathscr{F}\{\text{rect}\left(\frac{x}{1}\right)\} = \sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega}$$

Also,

$$\mathscr{F}\{\text{rect}\left(\frac{x}{2}\right)\} = \sqrt{\frac{2}{\pi}}\frac{\sin(2\omega)}{\omega}$$

Then, using the linearity of Fourier transform, the Fourier transform of the following functions can be computed as follow.

Some Properties of Fourier Transform



$$= \operatorname{rect}\left(\frac{x}{1}\right) + \operatorname{rect}\left(\frac{x}{2}\right)$$

Figure 8: Function $h(x)$

Therefore, the Fourier transform of $h(x)$ is

$$\begin{aligned}
\mathscr{F}\{h(x)\} &= \mathscr{F}\left\{\operatorname{rect}\left(\frac{x}{1}\right) + \operatorname{rect}\left(\frac{x}{2}\right)\right\} \\
&= \mathscr{F}\left\{\operatorname{rect}\left(\frac{x}{1}\right)\right\} + \mathscr{F}\left\{\operatorname{rect}\left(\frac{x}{2}\right)\right\} \\
&= \sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega} + \sqrt{\frac{2}{\pi}}\frac{\sin(2\omega)}{\omega} \qquad \blacksquare
\end{aligned}$$

Some Properties of Fourier Transform

Theorem (Fourier Transform of Derivative)

Let $f(x)$ be continuous function and $f'(x)$ is absolutely integrable. Then,

$$\mathscr{F}\{f'(x)\} = i\omega\mathscr{F}\{f(x)\}$$

Find the Fourier transform of $f(x) = xe^{-x^2}$

$$\mathscr{F}\{xe^{-x^2}\} = \mathscr{F}\{\underbrace{\frac{d}{dx}\left(-\frac{1}{2}e^{-x^2}\right)}_{xe^{-x^2}}\} = -\frac{1}{2}\cdot i\omega\cdot\mathscr{F}\{e^{-x^2}\}$$

$$= -\frac{1}{2}\cdot i\omega\cdot\frac{1}{\sqrt{2}}e^{-\omega^2/4}$$

$$= -\frac{\omega i}{2\sqrt{2}}e^{-\omega^2/4} \quad \blacksquare$$

Some Properties of Fourier Transform

Theorem (Convolution Theorem)

Suppose that $f(x)$ and $g(x)$ are piecewise continuous and absolutely integrable. Then,

$$\mathscr{F}\{f * g\} = \sqrt{2\pi}\,\mathscr{F}\{f(x)\}\,\mathscr{F}\{g(x)\}$$

Inverse Fourier Transform

Given Fourier transform $\hat{f}(\omega)$, we can also find $f(x)$ using (25).

Example FT-5: Find $f(x)$ of the following $\hat{f}(\omega)$

$$\hat{f}(\omega) = \begin{cases} 1, & -1 < \omega < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1) e^{i\omega x} d\omega = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i\omega x}}{ix} \right) \Big|_{-1}^{1} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{ix} - e^{-ix}}{ix} \right) \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{\sin(x)}{x} \right) \quad \blacksquare \end{aligned}$$

Application of Fourier Transform: Fourier Optics[1]

Fourier Optics

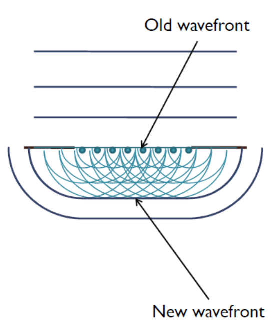


Figure 9: Illustration of new wavefront

From Physics, the basic of wave propagation, i.e., wave on surface water or light, the new wavefront can be generated by combining point sources located on the previous wavefront as shown in the figure.

[1] This example is based on the lecture note on Fundamentals of Optics by Dr.Waleed Soliman [3]

Application of Fourier Transform: Fourier Optics

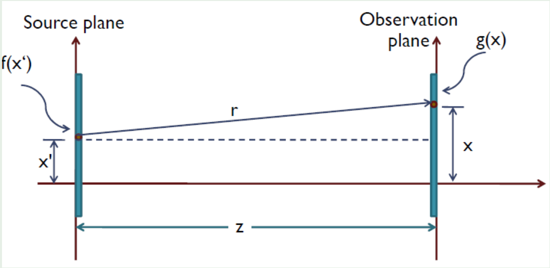


Figure 10: Source and Observation planes

Let's focus on the light source. Suppose we transmit light source, denoted by $f(x')$, through the very very small hole as in the figure. Based on the light propagation properties, the light that we can observe at the observation plane, denoted by $g(x)$, can be expressed by

$$g(x) = \frac{1}{r} f(x') e^{-i(\theta t - kr)}, \tag{26}$$

where, $k = \frac{2\pi}{\lambda}$ called *wave number* and $r = \sqrt{(x-x')^2 + z^2}$. However, the observation at point x should be a result due to all point sources from the source plane. Then, the observation at point x can be represented as the integral form.

$$g(x) = \frac{A}{\lambda} \int_{x'} \frac{1}{r} f(x') e^{-i(\theta t - kr)} dx' \tag{27}$$

Ignoring the fast varying time dependence term, $e^{-i\theta t}$, then, from (27)

$$g(x) = \frac{A}{\lambda} \int_{x'} \frac{1}{r} f(x') e^{ikr} dx', \tag{28}$$

where A is some constant. Now, consider

$$r = \sqrt{(x-x')^2 + z^2} = z\sqrt{1 + \frac{(x-x')^2}{z^2}}.$$

Application of Fourier Transform: Fourier Optics

By using Taylor series expansion, we have

$$kr = kz\sqrt{1+\frac{(x-x')^2}{z^2}} = kz\underbrace{\left(1+\frac{(x-x')^2}{2z^2}-\frac{(x-x')^4}{8z^4}+\dots\right)}_{\text{Taylor expansion of }\sqrt{1+\frac{(x-x')^2}{z^2}}}. \tag{29}$$

In (29), ignoring the higher terms in Taylor expansion (higher than second order), then (29) reduces to

$$kr = kz\left(1+\frac{(x-x')^2}{2z^2}\right) = kz + k\frac{(x-x')^2}{2z}. \tag{30}$$

Substitute (30) into (28), we get

Application of Fourier Transform: Fourier Optics

$$\begin{aligned} g(x) &= \frac{A}{\lambda}\int_{x'} \frac{1}{r} f(x') e^{i(kz + k\frac{(x-x')^2}{2z})} dx' \\ &= \frac{A}{\lambda}\int_{x'} \frac{1}{z} f(x') e^{i(kz + k\frac{(x-x')^2}{2z})} dx' \text{ (when } z >> (x-x')) \\ &= \frac{A\cdot e^{ikz}}{\lambda}\int_{x'} \frac{1}{z} f(x') e^{ik\frac{(x^2-2xx'+x'^2)}{2z}} dx' \\ &= \frac{A\cdot e^{ikz}\cdot e^{ikx^2}}{z\lambda}\int_{x'} f(x') e^{\frac{-ikx'^2}{2z}} e^{\frac{-ikxx'}{z}} dx' \end{aligned} \quad (31)$$

When z is very large, $\frac{kx'^2}{2z} \approx 0$. Also, since, $k = \frac{2\pi}{\lambda}$ and z are known, then

$$g(x) = \underbrace{\frac{A\cdot e^{ikz}\cdot e^{ikx^2}}{z\lambda}}_{\text{denoted as } \rho(x)} \cdot \int_{x'} f(x') e^{\frac{-ikxx'}{z}} dx'$$

Let's denote kx/z by ω_x. Finally, we arrive at the final equation.

$$g(x) = \underbrace{\rho(x)}_{\text{function of } x} \cdot \underbrace{\int_{x'} f(x') e^{-i\omega_x x'} dx'}_{\text{Fourier Transform of } f(x')} \quad \blacksquare \tag{32}$$

What we obtain from (32)? From (32), it tells us that **if we transmit the light source through a obstacle (think as a plate with some holes and let's denote the light pass through the obstacle by $f(x')$ as in the explanation above), then the light that we can observe at opposite plane (observation plane) will be the scaled fourier transform of $f(x')$ (scaling by factor $\rho(x)$).** It should exphasize that this statement will be true **when z is large in general.** However, it is possible to use *lens* to reduce the value of z.

Application of Fourier Transform: Power Spectral Density (Communication system)[3]

The *power spectral density (PSD)* or *power spectrum* describes the distribution of power of WSS[2] random process $X(t)$ as a function of frequency. The power spectrum of $X(t)$, denoted by $S_X(f)$, is the **Fourier Transform** of the autocorrelation function, denoted by $R_X(\tau)$, i.e.,

$$S_X(f) = \mathscr{F}[R_X(\tau)]$$

Now, how PSD and autocorrelation are related with communication system. One main application that PSD is significantly helped is the consideration of required transmission bandwidth of the digitally modulated signals.

[2]wide-sense stationary

[3]This section may be omitted since it assumes the prior knowledge of random process [**?**] and communication system [5]

Application of Fourier Transform: Power Spectral Density (Communication system)

We will calculate PSD of linearly modulated signals from three well-known modulation technique, Binary Phase Shift Keying (BPSK), Quadrature Phase Shift Keying (QPSK), and Offset Quadrature Phase-Shift Keying (OQPSK), and another modulattion technique, called Minimum Phase-Shift Keying, which provide better bandwidth efficiency over former three linear modulation techniques. In general, the PSD of linearly modulated signal is computed from complex envelope (or low pass equivalent) of such signal. The following calculation is based on [8] and [1].

Application of Fourier Transform: Power Spectral Density (Communication system)

Binary Phase Shift Keying (BPSK)

- Complex envelope of BPSK can be expressed as

$$v(t) = \sum_{n=0}^{\infty} I_n g(t - nT),$$

where I_n is information bit $\in \{-1, 1\}$ and $g(t)$ is rectangular puse shape of width T and amplitude A.

- Fourier transform of rectangular pulse $g(t)$ is

$$G(f) = \int_0^T A e^{-j2\pi ft} dt = A \left. \frac{e^{-j2\pi ft}}{-j2\pi f} \right|_0^T = A \left(\frac{1 - e^{-j2\pi fT}}{j2\pi f} \right)$$

$$= A e^{-j\pi fT} \left(\frac{e^{j\pi fT} - e^{-j\pi fT}}{j2\pi f} \right) = ATe^{-j\pi fT} \frac{\sin(\pi fT)}{\pi fT}$$

Application of Fourier Transform: Power Spectral Density (Communication system)

- Autocorrelation function of information sequence, $E[I_{n+m}I_n]$

$$E[I_{n+m}I_n] = \begin{cases} 1 \ , & m=0 \\ 0 \ , & m \neq 0 \end{cases}$$

- So, Fourier transform of $E[I_{n+m}I_n]$ is equal to

$$\Phi_{ii}(f) = \sum_{m=0}^{\infty} E[I_{n+m}I_n] e^{-j2\pi fmT} = 1$$

- Finally, PSD of BPSK can be shown as (for BPSK T = Tb)

$$\begin{aligned} \Phi_{vv}(f) &= \frac{1}{T}|G(f)|^2 \Phi_{ii}(f) \\ &= A^2 T \frac{\sin^2(\pi f T)}{\pi^2 f^2 T^2} \qquad \blacksquare. \end{aligned}$$

Application of Fourier Transform: Power Spectral Density (Communication system)

Quadrature Phase Shift Keying (QPSK)

- Complex envelope of QPSK can be expressed similar to BPSK but with complex value a_n, instead.

$$v(t) = \sum_{n=0}^{\infty} a_n g(t - nT),$$

where $a_n = b_n + jc_n$, bn and $cn \in \{-1, 1\}$, and $g(t)$ is rectangular puse shape of width T and amplitude A.

- So, $G(f)$ for QPSK is the same as BPSK

$$G(f) = ATe^{-j\pi fT} \frac{\sin(\pi fT)}{\pi fT}$$

(**Note :** T symbol period of QPSK is equal to 2Tb, Tb is bit period.)

Application of Fourier Transform: Power Spectral Density (Communication system)

- Autocorrelation function of information sequence, $E[I_{n+m}I_n]$

$$\begin{aligned} E[a_{n+m}{a^*}_n] &= E[(b_{n+m}+jc_{n+m})(b_n-jc_n)] \\ &= E[b_{n+m}b_n]-jE[b_{n+m}c_n]+jE[c_{n+m}b_n]+E[c_{n+m}c_n] \end{aligned}$$

- Since $E[b_n]=E[c_n]=(-1)\frac{1}{2}+(1)\frac{1}{2}=0$ and assume that component b_n and c_n are independent, so we get

$$E[a_{n+m}{a^*}_n] = \begin{cases} 1 & , \quad m=0 \\ 0 & , \quad m\neq 0 \end{cases}$$

- So, Fourier transform of $E[I_{n+m}I_n]$ is equal to

$$\Phi_{ii}(f) = \sum_{m=0}^{\infty} E[I_{n+m}I_n]e^{-j2\pi fmT} = 1$$

Application of Fourier Transform: Power Spectral Density (Communication system)

- Finally, PSD of QPSK can be shown as

$$\Phi_{vv}(f) = \frac{1}{T}|G(f)|^2\Phi_{ii}(f) = A^2T\frac{\sin^2(\pi f T)}{\pi^2 f^2 T^2}$$
$$= 2A^2T_b\frac{\sin^2(2\pi f T_b)}{4\pi^2 f^2 {T_b}^2} \quad ; \quad T = 2T_b \qquad \blacksquare.$$

Offset Quadrature Phase-Shift Keying (OQPSK)

OQPSK use the same pulse shape as QPSK. The only different is the phase change of OQPSK which is only $\pm\pi/2$. This can be done by delaying Quadrature component by half of symbol period. However, the delaying does not change PSD but it affects phase spectrum. So OQPSK has the same PSD as QPSK.

Application of Fourier Transform: Power Spectral Density (Communication system)

Minimum Phase-Shift Keying (MSK)

- We will derive PSD of MSK signal based on [2]. A cosine pulse of width T has Fourier transform

$$S(f) = -2T\frac{\cos(\pi fT)}{\pi(4f^2T^2-1)} = -4T_b\frac{\cos(2\pi fT_b)}{\pi(16f^2{T_b}^2-1)},$$

  where $T = 2T_b$.

- Since MSK can be written as

$$v(t) = \sum_{n=-\infty}^{\infty} a_n g(t-nT_b),$$

  where $a_n = \pm 1$ for n even and $a_n = \pm j$ for n odd, $g(t)$ is cosine pulse of width $T = 2T_b$.

Application of Fourier Transform: Power Spectral Density (Communication system)

- Hence, from [2], PSD of MSK can be shown as follow.

$$\Phi_{vv}(f) = \frac{1}{T_b}|S(f)|^2 = \frac{16A^2T_b}{\pi^2}\left(\frac{\cos(\pi f T)}{16f^2{T_b}^2 - 1}\right)^2 \quad \blacksquare.$$

Note that A is amplitude of cosine pulse.

The theoretical PSD based on the above calculation are plotted in Fig.11.

Application of Fourier Transform: Power Spectral Density (Communication system)

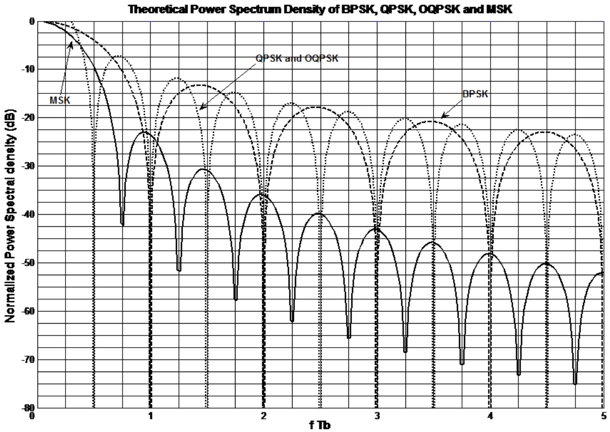


Figure 11: Theoretical Power Density of BPSK, QPSK, OQPSK, and MSK

Application of Fourier Transform: Power Spectral Density (Communication system)

From the figure, we can see from above figure that theoretically, BPSK has the largest mainlobe compared with QPSK, OQPSK and MSK. The width of mainlobe of BPSK is double of that of QPSK. So, it means BPSK signal requires larger BW to transmit compared with those three signaling. Mainlobe of MSK is between BPSK and QPSK/OQPSK. However, its sidelobe is pretty much lower than BPSK and QPSK/MSK. So, MSK might be suitable with narrow BW channel.

Comment on PSD

In practical, if the modulated signal is complicated, its PSD can be computed experimentally using MATLAB command: *pwelch*. *pwelch* is the MATLAB command that estimates power spectral density of signal sequence(sample) via Welchs method which is nonparametric power estimation method developed by Welch [5].

Summary

- The Fourier integral is the extension of Fourier series for non-periodic functions. The basic idea is to expand the period of periodic signal to infinity. Fourier integral is defined by (14)-(15).
- The definition of Fourier transform is derived from Fourier integral and can be computed by (24). The inverse version of Fourier transform is called inverse Fourier transform and can be copmuted by (25).
- The physical intepretation of the Fourier transform of $f(x)$ is considered as the superposition of all possible frequencies of sinusoidals functions that $f(x)$ is composed of. Fourier transform is also called *Spectral Density* and *Amplitute Spectrum*.
- In general, Fourier transform is complex-valued. Its magnitude and phase are called *Magnitude Spectrum* and *Phase Spectrum*, respectively.
- Fourier transform is widely used in many engineering applications such as optics, communications, signal processing, etc.

Find Fourier integral of the following functions (a is constant).

Exer 1. $f(x) = \begin{cases} 1 & ,0 < x < a \\ 0 & ,x > 0. \end{cases}$

Exer 2. $f(x) = \begin{cases} e^{-x} & ,0 < x < a \\ 0 & ,x > 0. \end{cases}$

Exer 3. $f(x) = \begin{cases} x & ,0 < x < 1 \\ 0 & ,x > 1. \end{cases}$

Exer 4. Compute Fourier integral of $f(x)$ in Exer 3. for $x = 1, 2,$ and 3 (say, compute $f(1)$,$f(2)$, and $f(3)$) when $a = 1, 2, 3, 4,$ and 5.

Find Fourier transform of the following functions (a and k are constant).

Exer 1. $$f(x) = \begin{cases} e^{kx} & , x < 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Exer 2. $$f(x) = \begin{cases} k & , 0 < x < a \\ 0 & , \text{otherwise.} \end{cases}$$

Exer 3. $$f(x) = \begin{cases} x & , -1 < x < 1 \\ 0 & , \text{otherwise.} \end{cases}$$

Exer 4. $f(x)$ is given as the following figure.

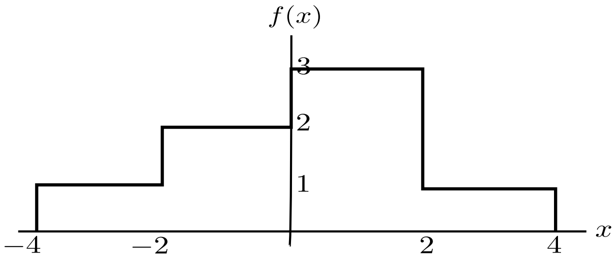


Figure 12: $f(x)$ in Exer 4.

Exer 5. Use result in Exer 3., plot *Magnitude Spectrum*