Pierre-Simon Laplace*

Laplace Transform

*Photo from *http:www.swlearning.comquantkohlerstatbiographical_sketchesbio8.1.html*

Laplace Transform

- Laplace transform is the useful method in solving initial-value problem involving linear, constant coefficient, ordinary and partial differential equations.

Definition (Laplace Transform)

Given $f(x)$. The Laplace transform of $f(x)$, denoted by $F(s)$, is defined as

$$F(s) = \mathscr{L}\{f(x)\} = \int_0^\infty f(x)e^{-sx}dx \tag{1}$$

Example LT-1: $f(x) = 1$, when $x \geq 0$. Compute $\mathscr{L}\{f(x)\}$

$$\mathscr{L}\{f(x)\} = \int_0^\infty (1)e^{-sx}dx = \left.\frac{e^{-sx}}{-s}\right|_0^\infty = -\frac{1}{2}(0-1)$$

$$= \frac{1}{s} \quad \blacksquare$$

Example LT-2: $f(x) = e^{ax}$, when $x \geq 0$, a is constant. Compute $\mathscr{L}\{f(x)\}$

$$\mathscr{L}\left\{e^{ax}\right\} = \int_0^\infty e^{ax} \cdot e^{-sx}dx = \int_0^\infty e^{-(s-a)x}dx$$

$$= \left.\frac{e^{-(s-a)x}}{-(s-a)}\right|_0^\infty$$

$$= -\frac{1}{s-a}(0-1) = \frac{1}{s-a} \quad \blacksquare$$

Example LT-3: Compute $\mathscr{L}\{\sin(ax)\}$

$$\begin{aligned}\mathscr{L}\{\sin(ax)\} &= \int_0^\infty \sin(ax)e^{-sx}dx \\ &= -\frac{e^{-sx}}{s^2+a^2}\left[s\sin(ax)+a\cos(ax)\right]\Big|_0^\infty \\ &= \frac{a}{s^2+a^2} \quad \blacksquare\end{aligned}$$

Try: find $\mathscr{L}\{\cos(ax)\}$

Answer: $\mathscr{L}\{\cos(ax)\} = \frac{s}{s^2+a^2}$

$f(x),\ x \geq 0$	$F(s)$
1	$\frac{1}{s}$
x	$\frac{1}{s^2}$
x^2	$\frac{2!}{s^3}$
$x^n,\ (n \geq 0)$	$\frac{n!}{s^{n+1}}$
e^{-ax}	$\frac{1}{s+a}$
$\cos(ax)$	$\frac{s}{s^2+a^2}$
$\sin(ax)$	$\frac{a}{s^2+a^2}$

$f(x),\ x \geq 0$	$F(s)$
$\cosh(ax)$	$\frac{s}{s^2-a^2}$
$\sinh(ax)$	$\frac{a}{s^2-a^2}$
$e^{-ax}\cos(bx)$	$\frac{s+a}{(s+a)^2-b^2}$
$e^{-ax}\sin(bx)$	$\frac{b}{(s+a)^2-b^2}$
$x^n e^{-ax},\ (n \geq 0)$	$\frac{n!}{(s+a)^{n+1}}$
x^a, a positive	$\frac{\Gamma(a+1)}{s^{a+1}}$

$\Gamma(..)$ is gamma function (See Appendix 3 in[1])

Table-1 The Laplace transform $F(s)$ of some basic functions $f(x)$

Heaviside Step Function

Definition (Heaviside Step Function)

$$u(x-a) = \begin{cases} 1 & ,x > a \\ 0 & ,x < a \end{cases} \tag{2}$$

Laplace transform of $u(x-a)$

$$\mathscr{L}\{u(x-a)\} = \int_a^{\infty} (1)e^{-sx}dx = \frac{e^{as}}{s}, \quad s > 0 \tag{3}$$

Example LT-4: Find $\mathscr{L}\{x\}$

$$\mathscr{L}\{t\} = \int_a^{\infty} xe^{-sx}dx = \left[-\frac{te^{-sx}}{s} - \frac{e^{-sx}}{s^2}\right]\Bigg|_0^{\infty}$$

$$= \frac{1}{s^2} \quad \blacksquare$$

Some Properties of Laplace Transform

Theorem (Linearity)

$$\mathscr{L}\{af(x)+bg(x)\} = a\mathscr{L}\{f(x)\} + b\mathscr{L}\{g(x)\}$$

a, b are constants.

Recall that

$$\mathscr{L}\{f(x)=1\} = \frac{1}{s}$$

$$\mathscr{L}\{f(x)=e^{ax}\} = \frac{1}{s-a}.$$

Try: find $\mathscr{L}\{x^n\}$

Answer: $\mathscr{L}\{x^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$

Some Properties of Laplace Transform

Example LT-5: Find $\mathscr{L}\{\cosh(ax)\}$

Recall that $\cosh(ax) = \frac{e^{ax}+e^{-ax}}{2}$

$$\begin{aligned}
\mathscr{L}\{\cosh(ax)\} &= \mathscr{L}\{\frac{1}{2}e^{ax} + \frac{1}{2}e^{-ax}\} \\
&= \frac{1}{2}\mathscr{L}\{e^{ax}\} + \frac{1}{2}\mathscr{L}\{e^{-ax}\} \\
&= \frac{1}{2}\left(\frac{1}{s-a}\right) + \frac{1}{2}\left(\frac{1}{s+a}\right) \\
&= \frac{s}{(s-a)(s+a)} \\
&= \frac{s}{s^2-a^2} \quad \blacksquare
\end{aligned}$$

Try: compute $\mathscr{L}\{\sinh(ax)\}$

Answer: $\mathscr{L}\{\sinh(ax)\} = \frac{a}{s^2-a^2}$

Example LT-6: Find $\mathscr{L}\{\sin(\omega x)\}$ (alternative method)

Recall that $\sin(\omega x) = \frac{e^{\omega x}+e^{-\omega x}}{2i}$

$$\begin{aligned}
\mathscr{L}\{\sin(\omega x)\} &= \mathscr{L}\{\frac{1}{2i}e^{-\omega x} - \frac{1}{2i}e^{-\omega x}\} \\
&= \frac{1}{2i}\mathscr{L}\{e^{\omega x}\} - \frac{1}{2i}\mathscr{L}\{e^{-\omega x}\} \\
&= \frac{1}{2i}\left(\frac{1}{s-\omega}\right) - \frac{1}{2i}\left(\frac{1}{s+\omega}\right) \\
&= \frac{\omega}{(s-\omega)(s+\omega)} \\
&= \frac{\omega}{s^2+\omega^2} \quad \blacksquare
\end{aligned}$$

Try: compute $\mathscr{L}\{\cos(\omega x)\}$ using an alternative method above

Answer: $\mathscr{L}\{\cos(\omega x)\} = \frac{s}{s^2+\omega^2}$

Some Properties of Laplace Transform

Theorem (Shift in S-domain)

If $\mathscr{L}\{f(x)\} = F(s)$*, then*

$$\mathscr{L}\{e^{ax}f(x)\} = F(s-a)$$

Example LT-7: Find $\mathscr{L}\{e^{ax}\sin(\omega x)\}$

since $\mathscr{L}\{\sin(\omega x)\} = \frac{\omega}{s^2+\omega^2}$, hence

$$\mathscr{L}\{e^{ax}\sin(\omega x)\} = \frac{\omega}{(s-a)^2+\omega^2}$$

Theorem (Laplace transform of Derivative)

If $\mathscr{L}\{f(x)\} = F(s)$*, then*

$$\mathscr{L}\{f'(x)\} = sF(s) - f(0)$$

$$\mathscr{L}\{f''(x)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}\{f^{(n)}(x)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$$

Some Properties of Laplace Transform

Example LT-8: Find $\mathscr{L}\{x\sin(\omega x)\}$

We know that

$$f(0) = 0, \quad f'(0) = [\omega x\cos(\omega x) + \sin(\omega x)]\,|_{x=0} = 0$$
$$f''(0) = \left[-\omega^2 x\sin(\omega x) + 2\omega\cos(\omega x)\right]|_{x=0} = 2\omega. \tag{4}$$

From theorem of Laplace transform of Derivative,

$$\mathscr{L}\{f''(x)\} = s^2\mathscr{L}\{f(x)\} - 0 - 0 \tag{5}$$

From (4), we also know that

$$\mathscr{L}\{f''(x)\} = -\omega^2\mathscr{L}\{f(x)\} + 2\omega\mathscr{L}\{\cos(\omega x\} \tag{6}$$

Since (5) = (6), hence

$$s^2\mathscr{L}\{f(x)\} = -\omega^2\mathscr{L}\{f(x)\} + 2\omega\frac{s}{s^2+\omega^2} \tag{7}$$
$$\mathscr{L}\{f(x)\} = \mathscr{L}\{x\sin(\omega x)\} = \frac{2\omega s}{(s^2+\omega^2)^2} \quad \blacksquare \tag{8}$$

Some Properties of Laplace Transform

Theorem (Laplace transform of integral of function)

If $\mathscr{L}\{f(x)\} = F(s)$, *then*

$$\mathscr{L}\left\{\int_0^x f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

Example LT-9: Find inverse Laplace transform of $G(s) = \frac{1}{s(s^2+\omega^2)}$

We already know that

$$\mathscr{L}\left\{\frac{\sin(\omega x)}{x}\right\} = \frac{1}{s^2+\omega^2}$$

Let's denote $\frac{1}{s^2+\omega^2}$ as $F(s)$. Then, by rearranging $G(s)$, we get the following relation.

$$\frac{1}{s(s^2+\omega^2)} = \frac{1}{s}\cdot\frac{1}{(s^2+\omega^2)} = \frac{F(s)}{s}$$

From Laplace transform of the integral of a function, we have

$$\frac{1}{s(s^2+\omega^2)} = \frac{F(s)}{s} = \mathscr{L}\left\{\int_0^x \frac{\sin(\omega x)}{x} d\tau\right\}$$

Hence, it appears that

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^x \frac{\sin(\omega x)}{x} d\tau$$
$$= \frac{1-\cos(\omega x)}{\omega} \quad \blacksquare$$

Try: Find inverse of $G(s) = \frac{1}{s^2(s^2+\omega^2)}$

Answer: $\mathscr{L}^{-1}\{G(s)\} = \frac{1-\cos(\omega x)}{\omega^2}$

Inverse Laplace Transform

Given Laplace transform $F(s)$, the inverse Laplace transform is denoted by

$$f(x) = \mathscr{L}^{-1}\{F(s)\}$$

In general, computing inverse Laplace transform directly from the definition of Laplace transform is very complicated [4]. Practically, we find inverse Laplace transform using

- Table of Laplace transform $\Rightarrow$ utilize formulas in the table.
- Partial fraction expansion $\Rightarrow$ use for Laplace transform, $F(s)$, in the rational form, i.e.,

$$F(s) = \frac{P(s)}{Q(s)},$$

  where $P(s)$ and $Q(s)$ are polynomial of s with the assumption that order of $P(s)$ is less than that of $Q(s)$.

Inverse Laplace Transform by Partial Fraction Expansion

- Most of Laplace functions we would deal with is **rational function**, i.e., $\frac{A(s)}{B(s)}$
- Most of rational functions can be expressed as the sum of simple rational function. For example,

$$F(s) = \frac{5s+13}{s^2+6s+5} = \frac{2}{s+1} + \frac{3}{s+5}$$

Note: degree of numerator must be lower than degree of denominator.

- Let $F(s) = \frac{P(s)}{Q(s)}$. If $Q(s)$ has degree n, it has n roots, i.e.,

$$\begin{aligned} Q(s) &= s^n + a_{n-1}s^{n-1} + \ldots + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0 \\ &= (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{n-1})(s-\alpha_n). \end{aligned}$$

Note: some α_j might be NOT distinct.

- Roots of $P(s)$ are called **"Zeros"** and roots of $Q(s)$ are called **"Poles"**

Inverse Laplace Transform by Partial Fraction Expansion

- After $F(s)$ is represented as partial fractional expression, the inverse Laplace transform can typically be done using the table of Laplace transform.
- In general, partial fraction expression can be done in three ways depending on the poles of $F(s)$ (roots of denominator $Q(s)$) as follow.

Inverse Laplace Transform by Partial Fraction Expansion

Case 1: Real and non-repeated roots ($Q(s)$)

$$F(s) = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \ldots + \frac{A_n}{s-\alpha_n}$$

where $\alpha_1, \ldots, \alpha_n$ are roots of $Q(s)$ and distinct. Let's try to find A_1

$$F(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-\alpha_1)Q_1(s)} = \frac{A_1}{s-\alpha_1} + \frac{P_1(s)}{Q_1(s)}, \quad (9)$$

where $Q_1(s)$ is a polynomial of order $n-1$ and

$$A_1 Q_1(s) + P_1(s)(s-\alpha_1) = P(s).$$

Multiply both side of (9) by $(s-\alpha_1)$. Then

Inverse Laplace Transform by Partial Fraction Expansion

$$(s-\alpha_1)F(s) = A_1 + (s-\alpha_1)\frac{P_1(s)}{Q_1(s)} = A_1 + \frac{P(s)}{Q_1(s)}$$

If we set $s = \alpha_1$ in (106), then we have

$$A_1 = \left.\frac{P(s)}{Q_1(s)}\right|_{s=\alpha_1}$$

For any other A_j, we can do similarly. Hence,

$$A_j = (s-\alpha_j)F(s)\big|_{s=\alpha_j}, \quad \blacksquare$$

for $j = 1, \ldots, n$.

Example LT-10: Compute partial fraction expansion of $F(s) = \frac{s^2+3s}{s^3+7s^2+14s+8}$

$$F(s) = \frac{s^2+3s}{s^3+7s^2+14s+8} = \frac{s^2+3s}{(s+1)(s+2)(s+4)}$$
$$= \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{A_3}{s+4}$$

where, using (106)

$$A_1 = \left.\frac{s^2+3s}{(s+2)(s+4)}\right|_{s=-1} = \frac{-2}{3}$$
$$A_2 = \left.\frac{s^2+3s}{(s+1)(s+4)}\right|_{s=-2} = \frac{-2}{-2} = 1$$
$$A_3 = \left.\frac{s^2+3s}{(s+1)(s+2)}\right|_{s=-4} = \frac{2}{3}$$

Inverse Laplace Transform by Partial Fraction Expansion

Try: Find inverse Laplace transform of $F(s) = \frac{s+1}{s^2+2s}$

Answer: $\frac{1}{2}\left(1+2e^{-2x}\right)u(x)$, where $u(x)$ is step function.

Case 2: Real and repeated roots ($Q(s)$)

$$F(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-\alpha_1)^{m_1}(s-\alpha_2)^{m_2}\cdots(s-\alpha_l)^{m_l}},$$

where $m_1 + m_2 + \ldots + m_l = n$ (recall that $Q(s)$ has order of n). For this case, $F(s)$ can be partially factor into the following form.

$$\begin{aligned} F(s) = {} & \frac{A_{(1,1)}}{(s-\alpha_1)} + \frac{A_{(1,2)}}{(s-\alpha_1)^2} + \ldots + \frac{A_{(1,m_1)}}{(s-\alpha_1)^{m_1}} \\ & + \frac{A_{(2,1)}}{(s-\alpha_2)} + \frac{A_{(2,2)}}{(s-\alpha_2)^2} + \ldots + \frac{A_{(2,m_2)}}{(s-\alpha_2)^{m_2}} \\ & + \ldots + \frac{A_{(l,1)}}{(s-\alpha_l)} + \frac{A_{(l,2)}}{(s-\alpha_l)^2} + \ldots + \frac{A_{(l,m_l)}}{(s-\alpha_l)^{m_l}} \end{aligned} \quad (10)$$

Inverse Laplace Transform by Partial Fraction Expansion

Let's try to find $A_{(1,m_1)}$. This can be done by mutiplying both sides by $(s-\alpha_l)^{m_1}$

$$(s-\alpha_l)^{m_1}F(s) = A_{(1,1)}(s-\alpha_1)^{m_1-1} + A_{(1,2)}(s-\alpha_1)^{m_1-2} + \dots$$
$$\dots + A_{(1,m_1)} + (s-\alpha_l)^{m_1}\,[\text{remaining terms}] \qquad (11)$$

Then, by setting $s=\alpha_1$ in (11), all terms in the right side of (11) become zero, except term $A_{(1,m_1)}$. Hence,

$$A_{(1,m_1)} = (s-\alpha_l)^{m_1}F(s)\big|_{s=\alpha_1} \qquad (12)$$

Now, let's try to find $A_{(1,m_1-1)}$. From (11), do derivative with respect to s for both sides.

$$\frac{d}{ds}\left[(s-\alpha_1)^{m_1}F(s)\right] = (m_1-1)A_{(1,1)}(s-\alpha_1)^{m_1-2} + (m_1-2)A_{(1,2)}(s-\alpha_1)^{m_1-3} + \ldots + A_{(1,m_1-1)} + \frac{d}{ds}\left\{(s-\alpha_1)^{m_1}\left[\text{remaining terms}\right]\right\} \quad (13)$$

Set $s = \alpha_1$ in (13). Then, similar to above, all terms in the right side of (13) become zero, except term $A_{(1,m_1-1)}$. Hence,

$$A_{(1,m_1-1)} = \frac{d}{ds}\left[(s-\alpha_1)^{m_1}F(s)\right]\Big|_{s=\alpha_1}$$

Inverse Laplace Transform by Partial Fraction Expansion

We can also find $A_{(1,m_1-2)}$ by similar way. More specifically, do derivative (13) with respect to s again and set $s = \alpha_1$. By rearranging the remainging term, we get

$$A_{(1,m_1-2)} = \frac{1}{2}\frac{d^2}{ds^2}\left[(s-\alpha_1)^{m_1}F(s)\right]\Big|_{s=\alpha_1}$$

In general, for any roots α_k, $k = 1,\ldots,l$, the corresponding constant $A_{(k,j)}$, for $j = 1,\ldots,m_k$, can be computed by

$$A_{(k,j)} = \frac{1}{(m_k-j)!}\frac{d^{(m_k-j)}}{ds^{(m_k-j)}}\left[(s-\alpha_k)^{m_k}F(s)\right]\Big|_{s=\alpha_k} \quad (14)$$

Example LT-11: Find partial fraction expansion of $F(s) = \frac{s}{(s+2)^2(s+1)}$

$$F(s) = \frac{s}{(s+2)^2(s+1)} = \frac{A_{(1,1)}}{(s+2)} + \frac{A_{(1,2)}}{(s+2)^2} + \frac{A_2}{(s+1)}$$

where,

$$A_2 = F(s)(s+1)|_{s=-1} = \left.\frac{s}{(s+2)^2}\right|_{s=-1} = -1$$

From (14),

$$\begin{aligned} A_{(1,1)} &= \frac{1}{(2-1)!}\frac{d^{(2-1)}}{ds^{(2-1)}}\left[(s+2)^2F(s)\right]\Big|_{s=-2} \\ &= \frac{d}{ds}\left[\frac{s}{(s+1)}\right]\Big|_{s=-2} = \left.\frac{(s+1)-s}{(s+1)^2}\right|_{s=-2} \\ &= 1 \end{aligned}$$

$$A_{(1,2)} = F(s)(s+2)^2\Big|_{s=-2} = \frac{s}{(s+1)}\Bigg|_{s=-2} = 2$$

Hence, we have

$$\frac{2}{(s+2)^2(s+1)} = \frac{1}{(s+2)} + \frac{2}{(s+2)^2} + \frac{-1}{(s+1)} \quad \blacksquare$$

Inverse Laplace Transform by Partial Fraction Expansion

Case 3: complex roots ($Q(s)$)

It is possible that some roots of $Q(s)$ (recall $F(s) = \frac{P(s)}{Q(s)}$) may be complex. In the polynomial with real coefficients, the roots will occur in complex conjugate pairs, for example, if there is root $\alpha_1 = -a + bi$, then there is alway another root $a_2 = -a - bi$. Hence, $F(s)$ will be in the following term.

$$
\begin{aligned}
F(s) = \frac{P(s)}{Q(s)} &= \frac{P(s)}{Q_1(s)(s+(a-bi))(s+(a+bi))} \\
&= \frac{P(s)}{Q_1(s)(s^2+2as+(a^2-b^2)} = \frac{P(s)}{Q_1(s)(s^2+\beta s+\gamma)},
\end{aligned}
$$

where $\beta = 2a$ and $\gamma = a^2 - b^2$.

Inverse Laplace Transform by Partial Fraction Expansion

Then, the corresponding partial fraction expansion of $F(s)$ can be expressed as

$$F(s) = \frac{P(s)}{Q_1(s)(s^2+\beta s+\gamma)} = \frac{As+B}{s^2+\beta s+\gamma} + \frac{P_1(s)}{Q_1(s)},$$

The constant A and B can be determined by clearing fraction and equating power of s. This may be clearly seen using the following example.

Example LT-12: Compute the partial fraction expansion of $F(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)}$

It is seen that the roots of $Q(s)$ are $-2, -1+i$, and $-1-i$. Hence,

$$F(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)} = \frac{As+B}{s^2+2s+2} + \frac{C}{s+2}. \tag{15}$$

Then, the constant C can be easily computed using (106).

$$\begin{aligned} C &= (s+2)F(s)|_{s=-2} = \left.\frac{2s^2+6s+6}{s^2+2s+2}\right|_{s=-2} \\ &= 1. \end{aligned}$$

The value A and B can be found by clearing fraction and equating the coefficients of power of s as follow. From (15),

Inverse Laplace Transform by Partial Fraction Expansion

$$\begin{aligned}\frac{2s^2+6s+6}{(s+2)(s^2+2s+2)} &= \frac{As+B}{s^2+2s+2}+\frac{1}{s+2}\\ &= \frac{(As+B)(s+2)+(s^2+2s+2)}{(s+2)(s^2+2s+2)}\\ &= \frac{(A+1)s^2+(2A+B+2)s+(2B+2)}{(s+2)(s^2+2s+2)}.\end{aligned}$$

Now, we have the following relations

$$\begin{aligned}A+1&=2\\ 2A+B+2&=6\\ 2B+2&=6\end{aligned}$$

Hence, we get $A = 1$ and $B = 2$. So, the partial fraction expansion of $F(s)$ is

$$F(s) = \frac{s+2}{s^2+2s+2} + \frac{1}{s+2} \qquad \blacksquare$$

In the case of repeated complex roots, we can apply the same approach as for repeated roots together with the method of clearing fraction and equating the power of s to find the constants corresponding to complex roots. Consider the next example.

Inverse Laplace Transform by Partial Fraction Expansion

Example LT-13: Find the inverse Laplace transform of $Y(s) = \frac{4s^2}{(s^2+1)^2(s+1)}$

The roots of $Q(s)$ (denominator of $Y(s)$) are $-i, -i$, and -1. $X(s)$ can be expressed as

$$Y(s) = \frac{4s^2}{(s^2+1)^2(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{(s^2+1)^2} + \frac{Ds+E}{s^2+1}$$

The constant A can be easily computed using the normal procedure of multiplying by $s+1$ and setting $s=-1$. So, $A=1$. The remaining constants can be found by clearing fraction and equating power of s as follow.

$$\frac{4s^2}{(s^2+1)^2(s+1)} = \frac{(s^2+1)^2 + (Bs+C)(s+1) + (Ds+E)(s^2+1)(s+1)}{(s^2+1)^2(s+1)}$$

We get the following relations

$$\begin{aligned} D+1=0 &\implies D=-1 \\ D+E=0 &\implies E=-D=1 \\ -2+B+D+E=0 &\implies B=2 \\ C+B+D+E=0 &\implies C=-B=-2 \\ 1+C+E=0 &\implies -1-2+1=0 \text{ (check)} \end{aligned}$$

Therefore,

$$Y(s)=\frac{1}{s+1}+\frac{2s-2}{(s^2+1)^2}+\frac{-s+1}{s^2+1}$$

Inverse Laplace Transform by Partial Fraction Expansion

By the linearity property of Laplace transfrom, the inverse Laplace transform can be done separately for each term in partial fraction expansion.

$$\begin{aligned}\mathscr{L}^{-1}\{Y(s)\} &= \mathscr{L}^{-1}\left\{\frac{1}{s+1}+\frac{2s-2}{(s^2+1)^2}+\frac{-s+1}{s^2+1}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathscr{L}^{-1}\left\{\frac{2s-2}{(s^2+1)^2}\right\}+\mathscr{L}^{-1}\left\{\frac{-s+1}{s^2+1}\right\}\end{aligned}$$

From the table of Laplace transform, $\mathscr{L}^{-1}\{X(s)\}$ is

$$\begin{aligned}y(x) &= \mathscr{L}^{-1}\{Y(s)\} \\ &= e^{-x}+(x\sin(x)+\sin(x)-x\cos(x))+(-\cos(x)+\sin(x)) \\ &= e^{-x}+(x+2)\sin(x)-(x+1)\cos(x) \qquad \blacksquare\end{aligned}$$

Laplace transform solving ODE and initial value problem

- Laplace transform is a method of solving ODE and initial value problem
- Differentiation of $f(x) \Longrightarrow$ multiplication of $\mathscr{L}\{f(x)\}$ by s
- Integration of $f(x) \Longrightarrow$ division of $\mathscr{L}\{f(x)\}$ by s

Laplace transform solving ODE and initial value problem

Consider the initial value problem

$$\begin{aligned} y'' + ay' + by &= r(x), \\ y(0) &= K_0, \\ y'(0) &= K_1, \end{aligned} \tag{16}$$

where y is a function of x, or $y(x)$. To solve such problem using Laplace transform, we can be divide into three main steps as follow.

Step 1: Transform (16) by taking Laplace transform

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + a\left[sY(s) - y(0)\right] + bY(s) = R(s), \tag{17}$$

where $R(s) = \mathscr{L}\{r(x)\}$. Note that to get (17), the laplace transform of derivative is used. Then, group term $Y(s)$ in (17) together.

$$\left(s^2 + as + b\right) Y(s) = (s+a)y(0) + y'(0) + R(s). \tag{18}$$

Step 2: Solving (18) by algebra

Divide (18) by s^2+as+b. Then, (18) becomes

$$Y(s) = \left[(s+a)y(0)+y'(0)\right]\cdot\frac{1}{s^2+as+b}+\frac{R(s)}{s^2+as+b}. \tag{19}$$

Let's denote $s^2+as+b=(s+\frac{1}{2}a)^2+b-\frac{1}{4}a^2$ by $\frac{1}{Q(s)}$. From (19),

$$Y(s) = \left[(s+a)y(0)+y'(0)\right]Q(s)+R(s)Q(s). \tag{20}$$

If $y(0)=y'(0)=0$, $Y(s)=R(s)Q(s)$, then

$$Q(s)=\frac{Y(s)}{R(s)}=\frac{\mathscr{L}\{output\}}{\mathscr{L}\{input\}} \tag{21}$$

Equation (21) is called **"Transfer Function"**.

Step 3: Inversion of $Y(s)$ in (20) to obtain the solution $y(x)$

$$y(x) = \mathscr{L}^{-1}\{Y(s)\}. \tag{22}$$

Note that we use second-order ODE in the above explanaion. However, for higher order ODE, the same principle can be applied.

Example LT-13: Solve $y''(x) - y(x) = x, \quad y(0) = 1, y'(0) = 1$

Step 1: Transform the problem by taking Laplace transform

$$\begin{aligned}
\mathscr{L}\{y''(x)\} - \mathscr{L}\{y(x)\} &= \mathscr{L}\{x\} \\
\left[s^2 Y(s) - sy(0) - y'(0)\right] - Y(s) &= \frac{1}{s^2} \\
(s^2-1)Y(s) &= \frac{1}{s^2} + s + 1
\end{aligned} \tag{23}$$

Step 2: Solve (23) for $Y(s)$ by algebra

$$\begin{aligned} Y(s) &= \frac{1}{s^2(s^2-1)} + \frac{s+1}{s^2-1} \\ &= \frac{1}{s^2(s^2-1)} + \frac{1}{s-1} \\ &= -\frac{1}{s^2} + \frac{1}{s^2-1} + \frac{1}{s-1} \end{aligned} \tag{24}$$

Step 3: Get $y(x)$ by inverse Laplace transform (24)

$$\begin{aligned} \mathscr{L}^{-1}\{Y(s)\} &= \mathscr{L}^{-1}\left\{-\frac{1}{s^2} + \frac{1}{s^2-1} + \frac{1}{s-1}\right\} \\ y(x) &= -x + sinh(x) + e^x \quad \blacksquare. \end{aligned}$$

Applications of Laplace Transform: Solution of Electrical Circuit

Solution of RLC electrical circuit

Given the RLC-circuit as follow.

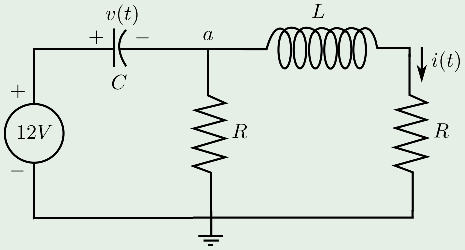

Figure 1: Given RLC circuit

Solve for the current $i(t)$ given $R = 1$ Ohm, $L = 1$ H, $C = 1$ F, and initial conditions are $v(0) = 4$ V and $i(0) = 2$ A.

Applications of Laplace Transform: Solution of Electrical Circuit

Apply nodal analysis at node a, we get

$$\frac{dv(t)}{dt} = \frac{12 - v(t)}{1} + i(t) \tag{25}$$

Also, apply mesh analysis at the right loop, we get another equation.

$$\frac{di(t)}{dt} + i(t) = \frac{12 - v(t)}{1} \tag{26}$$

Take Laplace transform both (25) and (26).

$$sV(s) - v(0) = \frac{12}{s} - V(s) + I(s) \tag{27}$$

$$sI(s) - i(0) + I(s) = \frac{12}{s} - V(s) \tag{28}$$

where $V(s)$ and $I(s)$ are the Laplace transform of $v(t)$ and $i(t)$, respectively.

Applications of Laplace Transform: Solution of Electrical Circuit

From (27) and (28), we can eliminate $V(s)$ and obtain only the equation in term of $I(s)$ as follow. First, from (27), we have the following relation.

$$V(s) = \frac{1}{s+1}\left(I(s) + v(0) + \frac{12}{s}\right) \tag{29}$$

Then, substitute (29) into (28) and rearrange the equation, we get

$$I(s) = \frac{2(s+5)}{s^2+s2+2} \tag{30}$$

Perform partial fraction expansion (30).

$$I(s) = \frac{2(s+1)}{(s+1)^2+1} + \frac{8}{(s+1)^2+1} \tag{31}$$

Finally, the solution $i(t)$ is obtained by inverse Laplace transform .

$$i(t) = e^{-t}\left(2\cos(t) + 8\sin(t)\right)\ A. \qquad \blacksquare.$$

Applications of Laplace Transform: Solution of Electrical Circuit

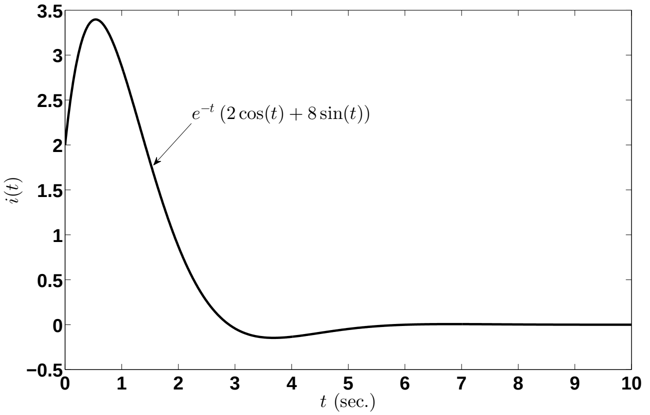

Figure 2: Current $i(t)$ over time

Applications of Laplace Transform: The Transformed Circuit

The transformed circuit

Similar to using phasor in circuit with sinusoidal excitation, using Laplace transform we can convert *R*, *L*, and *C* components to their transformed circuits in Laplace domain (or s-domain). Applying this techniques, we can solve the circuit algebraicly in s-domain, avoiding dealing with derivative. Also, the solution of this technique is complete solution which includes both natural and force responses. The equivalent circuits are shown in Fig. 3.

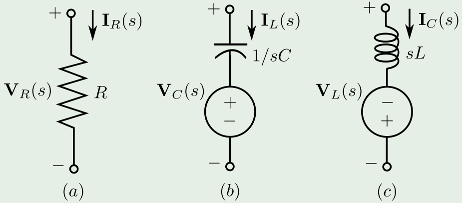

Figure 3: The transformed circuit: a) for resistor *R*, b) for inductor *L*, and c) for

Applications of Laplace Transform: The Transformed Circuit

To illustrate the benefit of this technique, consider the following circuit in Fig. 4a). Using the transformed circuit technique, it can be transformed into the circuit in s-domain as shown in Fig. 4b).

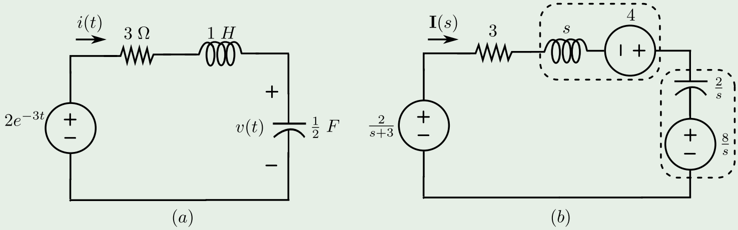

Figure 4: a) Circuit and b) Its transformed circuit

Applications of Laplace Transform: The Transformed Circuit

From Fig. 4b), using KVL we can show that

$$\mathbf{I}(s) = \frac{[2/(s+3)] + 4 - (8/s)}{3 + s + (2/s)}.$$

Then, apply partial fraction expansion, $\mathbf{I}(s)$ can be written as

$$\mathbf{I}(s) = \frac{13}{s+1} + \frac{20}{s+2} - \frac{3}{s+3}.$$

Finally, $i(t)$ can be obtained by taking inverse Z-transform $\mathbf{I}(s)$ in the above equation.

$$i(t) = -13e^{-t} + 20e^{-2t} - 3e^{-3t} \qquad \blacksquare.$$

Summary

- The Laplace transform is the important tool in solving initial value problem (IVP) which are widely found in many engineering problems. It is defined by (1).
- Using Laplace transform, the initial value problem is converted to algebra problem. Then, by solving the corresponding algebra problem, the solution of original initial value problem can be obtained by converting the solution of such algebra problem using inverse Laplace transform.
- Computing inverse Laplace transform directly from the definition is quite complicated. In general, it will be computed by table from Laplace transform or partial fraction expansion. The former is normally used if Laplace function is in term of basic functions such as sinusoidal functions, exponential functions, etc.

Summary

- Finding inverse Laplace transform via partial fraction expansion is most likely used since Laplace function is generally in form of rational function. With partial fractoin expansion, there will be three possible cases: 1) real and non-repeated roots, 2) real and repeated roots, and 3) complex-valued roots. Also, it is possible that such three cases may occur at the same time. After obtaining partial fraction expansion, inverse Laplace transfrom can be founed easily using table of Laplace transform.

Exercise (Laplace Transform)

In Exercise 1-4, find Laplace transform of the following functions (a, b are constant)

Exer 1. $x(t) = e^{-2t}$

Exer 2. $x(t) = te^{-at}$

Exer 3. $e^{2t}\sinh(t)$

Exer 4. $(a - bt)^2$

Exercise (Laplace Transform)

In Exercise 5-7, Find inverse Laplace transform using partial fraction expansion

Exer 5. $F(s) = \frac{5s+1}{s^2-25}$

Exer 6. $F(s) = \frac{s+10}{s^2-s-2}$

Exer 7. $F(s) = \frac{5s+1}{L^2s^2-n^2\pi^2}$, L and n are constant.