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Z-Transform

*Photo from WikiPedia

- A sequence $x[n]$, $-\infty < n < \infty$.

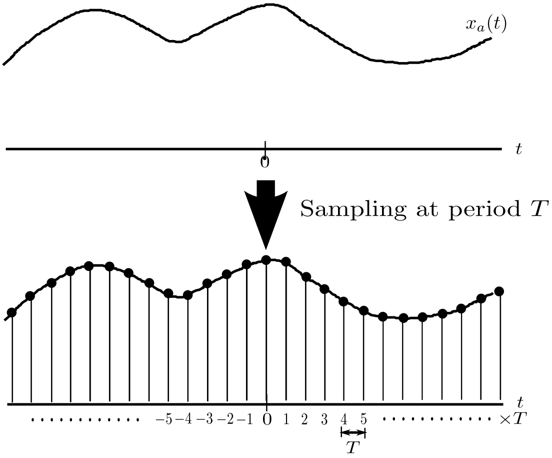
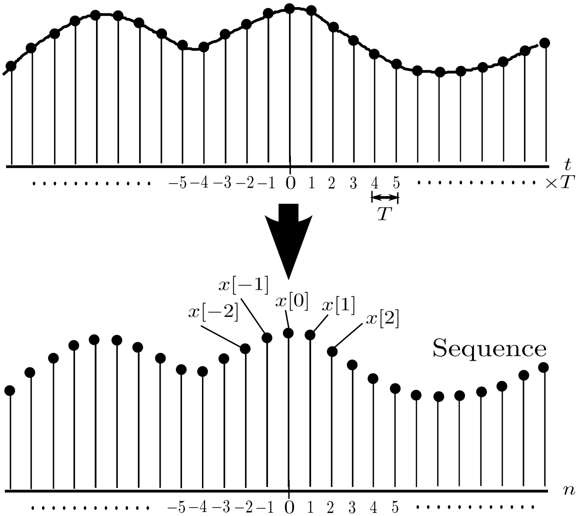


Figure 1: Sampling continuous signal $x_a(t)$

Z-Transform



$$x[n] = x_a(nT)$$

Figure 2: discrete sequence $x[n]$

Definition (Z-transform)

Given a sequence $x[n]$, for $n \geq 0$. The Z-transform of $x[n]$ is defined by[a]

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}, \qquad \text{(One-side Z-transform)} \tag{1}$$

$$\text{Notation:} \qquad x[n] \overset{\mathscr{Z}}{\Longleftrightarrow} X(z) \text{ or } \mathscr{Z}\{x[n]\} = X(z)$$

where z is a complex number. Since Z-transform is infinite summation, the set of z that $X(z)$ exists (finite value) need to be explicitly stated. Such set of all possible z is called **region of convergence (R.O.C)** and is required in almost every Z-transform computation.

[a] It should be noted that in this note we will focus only one-side Z-transform. The more general Z-transform is two-sided Z-transform which allows $x[n]$ to be defined for $n < 0$ is not considered here.

RECALL: Finite Geometric series: $$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \quad (2)$$

Infinite Geometric series: $$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1. \quad (3)$$

Example : Find Z-transform of the finite sequence $x_1[n] = \{1,2,5,7,0,1\}$, $x_2[n] = \delta[n]$, $x_3[n] = \delta[n-k]$, where $k > 0$

$$X_1(z) = \sum_{n=0}^{\infty} x_1[n]z^{-n} = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5},$$

R.O.C is all z-plane, except $z = 0$

$$X_2(z) = \sum_{n=0}^{\infty} x_2[n]z^{-n} = 1, \text{ R.O.C is all z-plane}$$

$$X_3(z) = \sum_{n=0}^{\infty} x_3[n]z^{-n} = z^{-k}, \text{ R.O.C is all z-plane, except } z = 0$$

Z-Transform

Example ZT-1: Find Z-transform of $x[n] = 2^n$

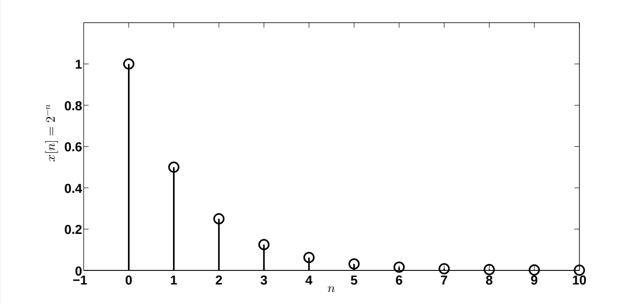


Figure 3: sequence $x[n]$ in Example ZT-1

$$X(z) = \sum_{n=0}^{\infty} 2^n \cdot z^{-n} = \sum_{n=0}^{\infty} (2z^{-1})^n$$

If $|2z^{-1}| < 1$, then the above serie exists. From (3),

$$\sum_{n=0}^{\infty}(2z^{-1})^n = \frac{1}{1-2z^{-1}}, \quad |2z^{-1}| < 1 \text{ or } |z| > 2.$$

$$2^n \overset{\mathscr{Z}}{\Longleftrightarrow} \frac{1}{1-2z^{-1}}, \quad |z| > 2 \quad \blacksquare$$

In general,

$$a^n \overset{\mathscr{Z}}{\Longleftrightarrow} \frac{1}{1-2z^{-1}}, \quad |z| > |a|. \tag{4}$$

It should be emphasized that the condition $|z| > |a|$ or R.O.C. are necessarily required since it will guarantee that the summation exists. The R.O.C can also be represented graphically as follow. Since $z = x + yi$ is complex (x and y are real), $|z| = \sqrt{x^2 + y^2}$. Hence, $|z| > |a|$ can be shown in the figure below.

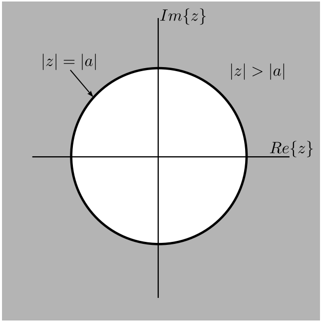


Figure 4: R.O.C. based on Z-transform in Eq.(4).

The shade region represents $|z| > |a|$. In general, the R.O.C can possibly be other region such as those shown in the following (shade) regions. However, since we only consider a sequence of $n \geq 0$ (causal sequence), so only R.O.C in the left figure is found in this note.

Z-Transform

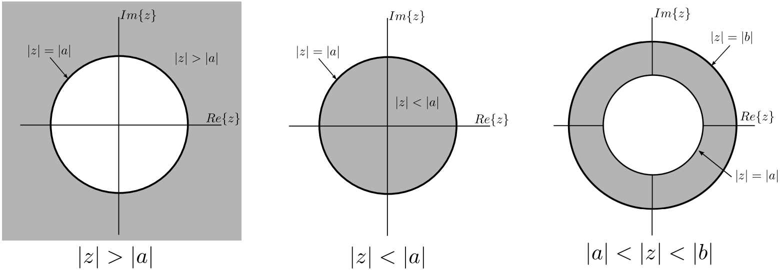


Figure 5: Possible R.O.C.

Example ZT-2: Show that $\mathscr{Z}\{(-\frac{1}{2})^n\} = \frac{2z}{2z+1}, |z| > \frac{1}{2}$

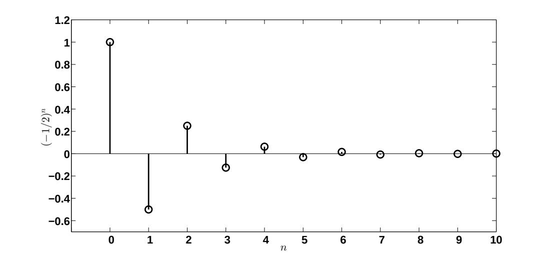


Figure 6: sequence $x[n]$ in Example ZT-1

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \cdot z^{-n} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}z^{-1}\right)^n$$

From (3),

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}z^{-1}\right)^{n} = \frac{1}{1-\left(-\frac{1}{2}z^{-1}\right)}, \quad |-\frac{1}{2}z^{-1}| < 1$$
$$= \frac{1}{1+\left(\frac{1}{2}z^{-1}\right)}, \quad |z| > |-\frac{1}{2}| = \frac{1}{2}$$
$$= \frac{2z}{2z+1}, \quad |z| > \frac{1}{2} \quad \blacksquare.$$

Some properties of Z-Transform

Linear property

Given sequences $x[n]$, $y[n]$ and constants a, b. Then,

$$\begin{aligned} \mathscr{Z}\{ax[n]+by[n]\} &= a\mathscr{Z}\{x[n]\}+b\mathscr{Z}\{y[n]\} \\ &= aX(z)+bY(z), \end{aligned} \tag{5}$$

where R.O.C is the intersection of each R.O.C.

Example ZT-3: What is $\mathscr{Z}\{\cos[\omega_0 n]\}$?

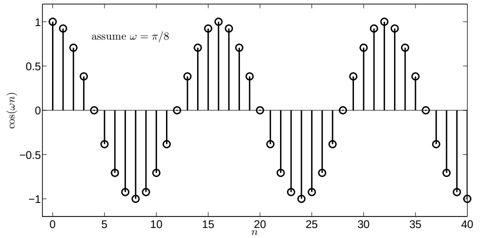


Figure 7: sequence $x[n] = \cos(\omega_0 n)$ in Example ZT-3

Some properties of Z-Transform

$$\mathscr{Z}\{\cos[\omega_0 n]\} = \sum_{n=0}^{\infty} \cos(\omega_0 n) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n} \right) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} e^{j\omega_0 n} z^{-n} + \sum_{n=0}^{\infty} \frac{1}{2} e^{-j\omega_0 n} z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{j\omega_0 n z^{-1}} \right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{-j\omega_0 n z^{-1}} \right)^n$$

From infinite geometric series in (3),

$$\mathscr{Z}\{\cos[\omega_0 n]\} = \frac{1}{2} \left(\frac{1}{1 - e^{j\omega_0 n} z^{-1}} \right) + \frac{1}{2} \left(\frac{1}{1 - e^{-j\omega_0 n} z^{-1}} \right),$$

$$\underbrace{|e^{j\omega_0 n} z^{-1}| < 1 \text{ and } |e^{-j\omega_0 n} z^{-1}| < 1}_{|z|>1}.$$

Some properties of Z-Transform

$$= \frac{1}{2}\left[\frac{1-e^{-j\omega_0 n}z^{-1}+1-e^{j\omega_0 n}z^{-1}}{(1-e^{j\omega_0 n}z^{-1})(1-e^{-j\omega_0 n}z^{-1})}\right], |z|>1$$

$$= \frac{1}{2}\left[\frac{2-2\cos(\omega_0 n)z^{-1}}{1-e^{-j\omega_0 n}z^{-1}-e^{j\omega_0 n}z^{-1}+z^{-2}}\right], |z|>1$$

$$= \frac{1-\cos[\omega_0 n]z^{-1}}{1-2\cos[\omega_0 n]z^{-1}+z^{-2}}, |z|>1.$$

$$\therefore \quad \cos[\omega_0 n] \overset{\mathscr{Z}}{\Longleftrightarrow} \frac{1-\cos[\omega_0 n]z^{-1}}{1-2\cos[(\omega_0 n]z^{-1}+z^{-2}}, |z|>1. \quad \blacksquare$$

Scaling property

If $x[n] \overset{\mathscr{Z}}{\Longleftrightarrow} X(z)$, R.O.C: $|z| > |a|$, then

$$r^n x[n] \overset{\mathscr{Z}}{\Longleftrightarrow} X(r^{-1}z), R.O.C : |z| > |r||a|. \tag{6}$$

where r is any real or complex number.

Example ZT-4: Compute the Z-transform of $x[n] = a^n \cos[\omega_0 n]$.

From previous example, we already know that

$$\cos[\omega_0 n] \overset{\mathscr{Z}}{\Longleftrightarrow} \frac{1 - \cos[\omega_0 n]z^{-1}}{1 - 2\cos[(\omega_0 n]z^{-1} + z^{-2}}, |z| > 1.$$

Then, from the scaling property of Z-transform, we get

$$a^n \cos[\omega_0 n] \overset{\mathscr{Z}}{\Longleftrightarrow} \frac{1 - a\cos[\omega_0 n]z^{-1}}{1 - 2a\cos[(\omega_0 n]z^{-1} + a^2 z^{-2}}, |z| > 1. \quad \blacksquare$$

Some properties of Z-Transform

Shift property (delaying)

Case 1: Time delay

If $x[n] \overset{\mathscr{Z}}{\Longleftrightarrow} X(z)$, then

$$x[n-n_0] \overset{\mathscr{Z}}{\Longleftrightarrow} z^{-n_0}\left[X(z)+\sum_{n=1}^{n_0} x[-n]z^{n}\right], \quad n_0>0. \tag{7}$$

When $x[n]$ is casual sequence, then

$$x[n-n_0] \overset{\mathscr{Z}}{\Longleftrightarrow} z^{-n_0}X(z). \tag{8}$$

Case 2: Time advance

If $x[n] \overset{\mathscr{Z}}{\Longleftrightarrow} X(z)$, then

$$x[n+n_0] \overset{\mathscr{Z}}{\Longleftrightarrow} z^{n_0}\left[X(z)-\sum_{n=0}^{n_0-1} x[n]z^{-n}\right], \quad n_0>0. \tag{9}$$

Some properties of Z-Transform

Example ZT-5: What is the Z-transform of $\left(\frac{1}{2}\right)^{n-2} u[n-2]$?

1. Using shift property, we already know that

$$x[n] = \left(\frac{1}{2}\right)^n \overset{\mathscr{Z}}{\Longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

By shift property of Z-transform, we get

$$x[n-2] = \left(\frac{1}{2}\right)^{n-2} \overset{\mathscr{Z}}{\Longleftrightarrow} z^{-2}X(z) = \frac{z^{-2}}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \quad \blacksquare$$

Note that later, the shifting property will be used when solving difference equations using Z-transform.

Some properties of Z-Transform

2. Direct calculation

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n-2} z^{-n} &= \left(\frac{1}{2}\right)^{-2} \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n} z^{-n} = 4\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^{n} z^{-n} \\
&= 4\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n} z^{-n} - 1 - \frac{1}{2}z^{-1}\right) \\
&= 4\left(\frac{1}{1-\frac{1}{2}z^{-1}} - 1 - \frac{1}{2}z^{-1}\right), \quad |\frac{1}{2}z^{-1}| < 1 \\
&= 4\left(\frac{\frac{1}{4}z^{-2}}{1-\frac{1}{2}z^{-1}}\right), \quad |z| > \frac{1}{2} \\
&= \frac{z^{-2}}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \quad \blacksquare
\end{aligned}
$$

Inverse Z-Transform

- The inverse Z-transform is formally given by

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz, \qquad (10)$$

  where C can be simply taken as a circle in the R.O.C of X(z) in z-plane. Because the contour integral in (10) is difficult to compute, in practical other more simple methods are used. Such methods are 1) Power series expansion, 2) Recursion[4] and 3) Partial fraction expansion. In this note, we only consider the first and third methods.

- Now, we would consider inverse Z-transform of rational function which is generally found in most engineering applications.

$$\frac{P(z)}{Q(z)} \Longrightarrow \text{similar to inverse Laplace transform.}$$

[4] Jenkins, L.B., 1967: A useful recursive form for obtaining inverse z-transforms. Proc. IEEE, 55, 574575. IEEE.

Inverse Z-Transform

Inverse Z-transform by power series expansion

Given X(z) with its corresponding R.O.C, we will try to rewrite X(z) in the the following expression using long-division process.

$$X(z) = \sum_{n=0}^{\infty} c_n z^{-n} = c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \dots \qquad (11)$$

Then, by the definition of Z-transform, $x[n] = c_n$, for all n. Also, from the above expression, $x[n]$ can possibly be reduced into single expression.

The long-division of rational function is demonstrated in the next examples.

Example ZT-6: Find the inverse Z-transform of
$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}, |z| > 1$

$$
\begin{array}{rl}
 & 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \quad \ldots \\
\hline
1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} \ \Big) & 1 \\
 & 1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-1} \\
\hline
 & \frac{3}{2}z^{-1} - \frac{1}{2}z^{-1} \\
 & \frac{3}{2}z^{-1} - \frac{9}{4}z^{-2} \quad \frac{3}{4}z^{-3} \\
\hline
 & \frac{7}{4}z^{-2} - \frac{3}{4}z^{-3} \\
 & \frac{7}{4}z^{-2} - \frac{21}{8}z^{-3} \quad \frac{7}{8}z^{-4} \\
\hline
 & \frac{15}{8}z^{-3} - \frac{7}{8}z^{-4} \\
 & \frac{15}{8}z^{-3} - \frac{45}{16}z^{-4} \quad \frac{15}{16}z^{-5} \\
\hline
 & \frac{31}{16}z^{-4} - \frac{15}{16}z^{-5} \\
 & \vdots \qquad \vdots
\end{array}
$$

Inverse Z-Transform

Thus, from the definition of Z-transfrom, we have

$$x[n] = \left\{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots\right\}$$
$$= \frac{2^{(n+1)} - 1}{2^n}, n \geq 0 \quad \blacksquare$$

Try: Find the inverse Z-transform of $X(z) = \frac{3-\frac{5}{2}z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-1}}$ using power series expansion

Answer: $x[n] = \{3, 2, \frac{3}{2}, \frac{5}{4}, \ldots\}$

Inverse Z-transform by partial fraction expansion

Given that X(z) is in the following rational form.

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-n}} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}, \tag{12}$$

where $M < N$ which means that $X(z)$ is *proper* rational function. For improper rational function $(M \geq N)$, we can perform long-division and express such rational function as a polynomial and a proper rational function. The following example will demonstrated this operation. It should also be noted that $X(z)$ in (12) has $a_0 = 1$. If $a_0 \neq 0$, we can obtain the form in (12) by dividing both numerator and denominator by a_0.

Inverse Z-Transform

Then, dividing both a numerator and a denominator in (12) by z^N.

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \ldots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \ldots + a_N}.$$

Rearranging $X(z)$ to the following form.

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \ldots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \ldots + a_N} \tag{13}$$

Since $N > M$, (13) is always proper. Now, we can do partial fraction expansion the right side of (13) using the same principle as we do in Laplace transform (only change variable s to z). When partial fraction expansion is obtained together with a given R.O.C. and table of Z-transform, the inverse Z-transform can be found.

Example ZT-7: Find inverse Z-transform of
$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}, |z| > 2$

Let's elimiate term z^{-2} by dividing both numerator and denominator by z^{-2}.

$$X(z) = \frac{z^2}{z^2 - 1.5z^1 + 0.5}$$

Rearranging X(z) as the form in (13) and perform the expansion. We obtain

$$\frac{X(z)}{z} = \frac{z}{z^2 - 1.5z^1 + 0.5} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}.$$

Notice that in this case, all poles of $X(z)$ are distinct. So, the constant A_1 and A_2 can be computed as follow.

$$A_1 = \frac{X(z)}{z}(z-1)\bigg|_{z=1} = \frac{z}{z-0.5}\bigg|_{z=1} = 2.$$

$$A_2 = \frac{X(z)}{z}(z-2)\bigg|_{z=0.5} = \frac{z}{z-1}\bigg|_{z=0.5} = 1.$$

Now, we have

$$X(z) = z\left(\frac{2}{z-1} + \frac{1}{z-0.5}\right) = \frac{2}{1-z^{-1}} + \frac{1}{1-0.5z^{-1}}$$

Hence, since R.O.C is $|z| > 2$, inverse Z-transform of $X(z)$ is

$$x[n] = \mathscr{Z}^{-1}\left\{\frac{2}{1-z^{-1}} + \frac{1}{1-0.5z^{-1}}\right\} = 2u[n] + (0.5)^n u[n], \quad \blacksquare$$

where $u[n]$ is unit-step function.

Example ZT-8: What is $\mathscr{Z}^{-1}\left\{\frac{1}{(1+z^{-1})(1-z^{-1})^2}\right\}, |z| > 1.$

First, let a given rational function be denoted by $X(z)$. Then, dividing both numerator and denominator of $X(z)$ by z^3 and rearranging the result as the form in (13)

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

In this case, there is one repeated poles. So, as in Laplace transform, the partial fraction expansion of $X(z)$ is

$$\frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_{2,1}}{z-1} + \frac{A_{2,2}}{(z-1)^2}$$

Inverse Z-transform

where

$$A_1 = \frac{X(z)}{z}(z+1)\bigg|_{z=-1} = \frac{z^2}{(z-1)^2}\bigg|_{z=-1} = \frac{1}{4}.$$

$$A_{2,2} = \frac{X(z)}{z}(z-1)^2\bigg|_{z=1} = \frac{z^2}{(z+1)}\bigg|_{z=1} = \frac{1}{2}.$$

$$A_{2,1} = \frac{d}{dz}\left[\frac{X(z)}{z}(z-1)^2\right]\bigg|_{z=1}$$

$$= \frac{d}{dz}\left[\frac{z^2}{(z+1)}\right]\bigg|_{z=1} = \frac{3}{4}.$$

Now, rearranging the equation and, hence, the inverse Z-transform of $X(z)$ is

Inverse Z-transform

$$x[n] = \mathscr{Z}^{-1}\left\{\frac{\frac{1}{4}}{1+z^{-1}} + \frac{\frac{3}{4}}{1-z^{-1}} + \frac{\frac{1}{2}z^{-1}}{(1-z^{-1})^2}\right\}$$
$$= \frac{1}{4}(-1)^n u[n] + \frac{3}{4}u[n] + \frac{1}{2}nu[n] \quad \blacksquare.$$

Solution of Difference Equations

- Similar idea as differential equations in continuous-time systems. But difference equations is for discrete-time system.
- The general form of difference equations is as follow.

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \tag{14}$$

- To solve difference equations using Z-transform, the similar steps as in solving differential equations using Laplace transform can be done with the use of the shifting property of Z-transform.

Consider the following example for the demonstration of using Z-transform in solving difference equations.

Example ZT-9: Try to solve the following difference equation using Z-transform

$$y[n] - \alpha y[n-1] = u[n], \quad -1 < \alpha < 1,$$

where the initial condition $y[-1] = 1$.

Taking Z-transform both sides of the equation and using (7), we get

$$Y(z) - \alpha\left(z^{-1}Y(z) + y[-1]\right) = X(z)$$

Rearraning the equation.

$$Y(z)\left(1 - \alpha z^{-1}\right) = \alpha + \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{1}{(1 - \alpha z^{-1})(1 - z^{-1})}$$

Performing partial fraction expansion the second term in the right side of the above equation, now we have

$$\begin{aligned} Y(z) &= \frac{\alpha}{1-\alpha z^{-1}} + \frac{1}{1-\alpha}\left(\frac{1}{1-z^{-1}} - \frac{\alpha}{1-\alpha z^{-1}}\right) \\ &= -\frac{\alpha^2}{1-\alpha}\left(\frac{1}{1-\alpha z^{-1}}\right) + \frac{1}{1-\alpha}\left(\frac{1}{1-z^{-1}}\right) \end{aligned}$$

Finally, do inverse Z-transform to obtain the solution $y[n]$.

$$\begin{aligned} y[n] &= \left(-\frac{\alpha^2}{1-\alpha}\right)\alpha^n u[n] + \left(\frac{1}{1-\alpha}\right)u[n] \\ &= \left(\frac{1-\alpha^{n+2}}{1-\alpha}\right)u[n] \quad \blacksquare \end{aligned}$$

Analysis of digital filter

A basic lowpass digital filter is implemented based on regular direct form II structure (see [5] for more detail) as shown in the following figure.

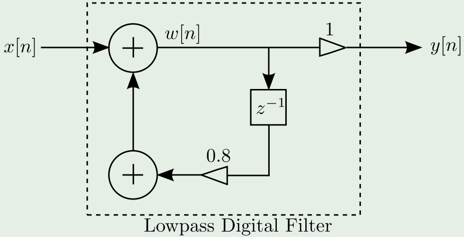


Figure 8: Basic lowpass digital filter

where $x[n]$ and $y[n]$ are input and output sequences, respectively. Find the step response of given digital filter with zero initial condition.

From the figure, we have two relations.

$$y[n] = w[n] \tag{15}$$

$$w[n] = 0.8w[n-1] + x[n] \tag{16}$$

Then, from (16) we get the following difference equation

$$y[n] = 0.8y[n-1] + x[n] \tag{17}$$

Take Z-transform both sides of the equation and use (7).

$$Y(z) = 0.8z^{-1}Y(z) + \frac{1}{1-z^{-1}} \quad (x[n] = u[n])$$

Rearrange $Y(z)$ and then do partial fraction expansion of $Y(z)$

$$Y(z) = \frac{1}{(1-z^{-1})(1-0.8z^{-1})} = \frac{0.5}{1-z^{-1}} + \frac{-0.5}{1-0.8z^{-1}}$$

Finally, take inverse Z-transform to obtain $y[n]$.

$$y[n] = 0.5u[n] - 0.5(0.8)^n u[n] \qquad \blacksquare.$$

Application of Z-transform: Digital Filter

Echo Cancellation Filter

Echo signal may possibly be considered as the summation of original signal and its distorted and delayed version. The example of echo signal is the echos of the audio signal in concert hall. Based on this consideration, echo signal can be expressed mathematically as,

$$y[n] = x[n] + \sum_{i=0}^{K} \alpha_i x[n - n_i] \quad \stackrel{\mathscr{Z}}{\Longleftrightarrow} \quad Y(z) = (1 + \sum_{i=0}^{K} \alpha_i z^{n_i})X(z) \tag{18}$$

where $x(n)$ is the original signal, n_i is the amount of delay in samples of echo and α_i is the relative strength, for $i = 1, \ldots, K$. Take Z-transform the above equation, we can consider that $y(n)$ is the output of the system with transfer function $H(z)$ as shown in the following figure.

Application of Z-transform: Digital Filter

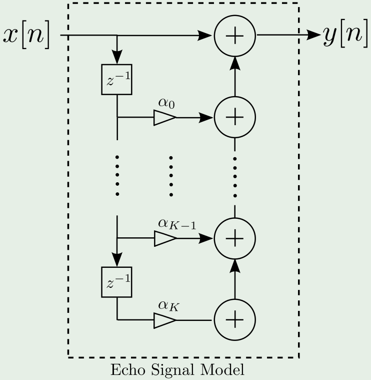


Figure 9: Echo signal model $H(z)$

Suppose that we know all values of α_i and n_i. We can generate *echo-free* signal by filtering $y[n]$ with filter in Fig. 10.

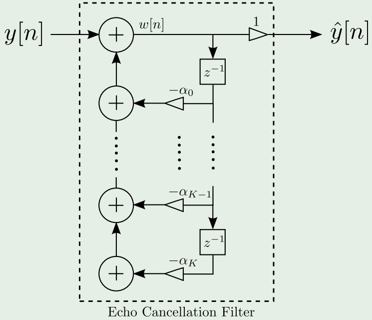


Figure 10: Echo Cancellation Filter

The output of the above filter, $\hat{y}[n]$, will be exactly $x[n]$. This can be shown as follow. From the Fig. 10, we know that

Application of Z-transform: Digital Filter

$$\hat{y}[n] = w[n]$$

$$w[n] = y[n] - \sum_{i=0}^{K} \alpha_i w[n-n_i]$$

From the above two relations, we can express $\hat{y}[n]$ as

$$\hat{y}[n] = y[n] - \sum_{i=0}^{K} \alpha_i \hat{y}[n-n_i]$$

Take Z-transform both sides, we have

$$\hat{Y}(z) = Y(z) - \sum_{i=0}^{K} \alpha_i z^{n_i} \hat{Y}(z) \quad \Longrightarrow \quad \hat{Y}(z) = \frac{1}{1+\sum_{i=0}^{K} \alpha_i z^{n_i}} Y(z) \qquad (19)$$

Application of Z-transform: Digital Filter

$$\hat{Y}(z) = \left[\frac{1}{1+\sum_{i=0}^{K}\alpha_i z^{n_i}}\right](1+\sum_{i=0}^{K}\alpha_i z^{n_i})X(z) = X[n]$$

As the result, we get $x[n]$ back. This particular filter in Fig. 10 is called *echo cancellation filter*. The crucial point to construct echo cancellation filter is that all values of α_i and n_i must be known in advance. This condition is NOT quite possible in real environment. Moreover, in reality, the value of α_i and n_i are also varied in time (or, mathematically, $\alpha_i(t)$ and $n_i(t)$). This makes much more difficulty in constructing the effective echo cancellation filter.

Application of Z-transform: Digital Filter

Comment on Echo Cancellation Fitler

In (18), if we know that echo signal $y[n]$ is just summation of original signal $x[n]$ and a few distorted and delayed versions of $x[n]$ ($K \approx 1$ or 2), it may be possible to simply estimate exact or almost exact α_i and n_i using autocorrelation technique. See [**?**] for more detail. In case of α_i and n_i are varied in time, this technique may be applied together with windowing operation of echo signal. We should also emphasize that absolutely there exist many more complicated techniques to construct echo cancellation filter, but autocorrelation technique is quite simple and very effective for such particular case.

Summary

- Similar to Laplace transform for continuous functions/signals, Z-transform is the transform technique for discrete functions/signals. Its definition is defined by (1).
- Since Z-transform is in term of infinite summation, Z-transform is properly defined with the set of all possible z that give finite value of the summation. Such set of z is called *region of convergence (R.O.C.)*.
- Z-transform is useful tool in solving engineering problems related to difference equations of discrete-time system. Like Laplace transform, Z-transform converts difference equations to algebra equations. Then, the solution of difference equations can be obtained by converting the solution of corresponding algebra equations using inverse Z-transform.
- Direct computing inverse Z-transform from the definition is really difficult. In general, there are other simple methods can be used such as power series expansion, recursion, and pratial fraction expansion. The partial fraction expansion is commonly used since most of the Z-transform is rational functions which are found in various engineering problems.

In Exercise 1-3, find Z-transform of the following $x[n]$.

Exer 1. $$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & ,n \geq 5 \\ 0 & ,n \leq 4. \end{cases}$$

Exer 2. $x[n]$ is given in the figure below

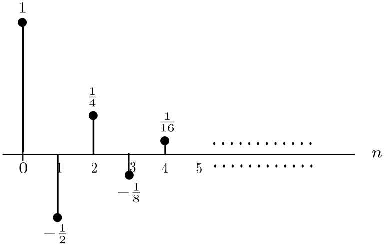


Figure 8: sequence $x[n]$ for Exer 2.

Exercise (Z-Transform)

Exer 3. $x[n] = a^n \cos(\omega_0 n, \quad n > 0, \omega_0 > 0$

In Exercise 4-5, find $x[n]$ from the following $X(z)$.

Exer 4. $X(z) = \frac{1-2z^{-1}+2z^{-2}-z^{-3}}{(1-z^{-1})(1-0.5z^{-1})(1-0.2z^{-1})}$

Exer 5. $X(z) = \frac{2-1.5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}$

Exer 6. Find $y[n]$ from the following difference equation.

$$y[n+2] - 5y[n+1] + 6y[n] = \left(\frac{1}{2}\right)^n,$$

where $y[0] = y[1] = 0$.