Unit One:

Electrostatics: Coulomb's law

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1a. Coulomb's law

Charges and force

An electrical charge is defined as a property of matter that causes it to experience a force when exposed to an electrical field.

So, an electrical charge is a property of matter. As we might all know, charges can be either positive (such as the case for protons) or negative (such as the case of electrons). We might have as well been taught that like charges attract each other and opposite charges repel each other.

When we say that a charge attracts the opposite or repels the like, we understand that these charges experience a force that either pushes them towards each other or pushes them away as shown by the arrows in figure 1.1. In either case, the force on each charge is pointing in the opposite direction.

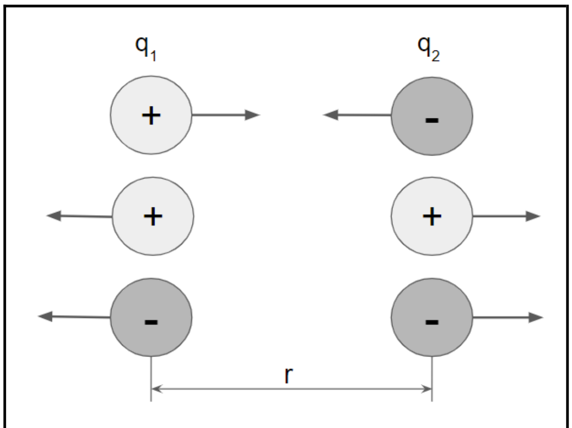

Figure 1.1: Like charges repel each other while opposite charges attract each other.

When those charges are at rest, this force follows an experimental law known as Coulomb's law. A law that is named after a French physicist Charles-Augustin de Coulomb who published it in 1785. Similarto Newton's gravitational force, the force of attraction or repelling between the two charges is inversely proportional to the square of the distance between them. Mathematically the force is

$$\overline{F} = k_e \frac{q_1 q_2}{r^2} \hat{r} \qquad (1.1)$$

Here, q_1 and q_2 are the magnitude of the charges with the proper sign (- for negative charge and + for positive charge). Charge magnitude has a unit of Coulomb or C in short. Originally 1 C was the amount of charges received by an electrical current of 1 Amp in 1 second. Hence, one electron (or one proton) carries an elementary charge amplitude of

$$q_e = 1.602176634 \times 10 - 19\ C \qquad (1.2)$$

For the force in equation 1.1, the constant k_e is the Coulomb's constant and it has a magnitude of

$$k_e \approx 8.988 \times 109\ N \cdot m^2 \cdot C^{-2} \qquad (1.3)$$

Vector force

Notice that in equation 1.1 the force is represented as a vector. The direction of the vector depends on unit vector $\hat{r}$ and the sign of the product of the two charges q_1q_2. We understand how to properly assign the signs of the charges. The question is which direction does this unit point towards. The answer depends which force we are considering, the one applied on q_1 or on q_2.

For a force created by q_1 and exerted on q_2, the unit vector $\hat{r}$ is pointing from q_1 to q_2, while the force created by q_2 and exerted on q_1, the unit vector $\hat{r}$ is pointing from q_2 to q_1. We can write that mathematically as

$$\overline{F}_1 = q_2 \cdot \left(k_e \frac{q_1}{r^2} \hat{r}_{12}\right) \tag{1.4}$$

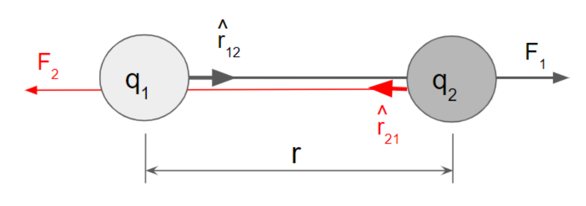

Figure 1.2: Vectorial force representation..

Similarly, the force on q_1 due to q_2 is

$$\overline{F}_2 = q_1 \cdot \left(k_e \frac{q_2}{r^2} \hat{r}_{21}\right) \tag{1.5}$$

From equations 1.4 and 1.5 one can say that the force exerted on a charge is the multiplication of the charge amplitude q by some vector quantity (the terms in parentheses in equation 1.2 and 1.3). This vector quantity is known as the electric field.

Electric field

One way to look at how one charge affects another is to read Coulomb's law differently. We say that for instance, charge q_1 emits an electric field that points in the radial direction and when a charge q_2 is placed in this field it experiences a force F_1 as in equation 1.4. In other words

$$\overline{F}_1 = q_2 \cdot \overline{E}_1 \tag{1.6}$$

$$\overline{E}_1 = k_e \frac{q_1}{r^2} \hat{r}_{12} \tag{1.7}$$

According to the graph in figure 1.2 and equation 1.7, the sign of the charge amplitude q_1 determines if the electric field is pointing outwards if q_1 is positive or downwards if q_1 is negative as illustrated in figure 1.3. .So in general, a static charge q emits an electric field which amplitude at any distance r from the charge is

$$\overline{E} = k_e \frac{q}{r^2} \hat{r} \tag{1.8}$$

Where the unit vector $\hat{r}$ is in the radial direction.

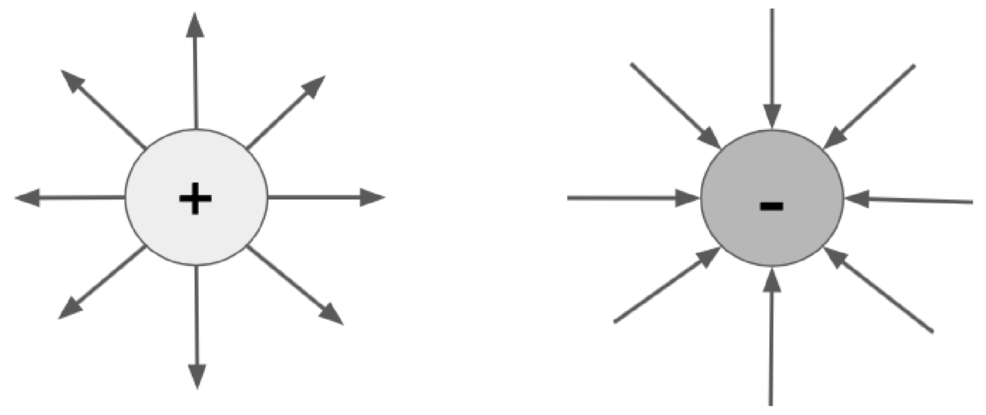

Figure 1.3: Direction of the electric field emitting from one charge.

From equation 1.6 we can tell that the unit of the electric field is force/charge or N/C.

Section review questions

1a-1) Complete the table

Point 1 (m)	Point 2 (m)	q_1 (C)	q_2 (c)	F_1
$(10^{-6},0,0)$	$(-10^{-6},0,0)$	$5x10^{-15}$	$-5x10^{-15}$	
$(0,0,0)$	$(10^{-6},10^{-6},0)$	$5x10^{-15}$	$5x10^{-15}$	
$(0,10^{-6},2x10^{-6})$	$(0,10^{-6},5x10^{-6})$	$2x10^{-15}$	$-4x10^{-15}$	

1a-2) A charge q1 tha has 2 positive elementary charges amplitude is placed at the origin of the coordinates. It experiences a force of $4x10^{-12}$ N pushing it towards the positive x-direction.

(a) What is the amplitude of the electric field applied on the charge.
(b) If the charge causing the electric field is placed on the negative side of the x-axis what sign would be the charge: i. Positive ii. Negative
(c) If the charge causing the field has an amplitude of four elementary charges, what is the distance between the two charges?
(d) If the charge causing the field is now moved to the positive x-axis, what direction would be the applied force?

1a-3) Write an expression for the electric field emitted by a charge of amplitude $2x10^{-14}$ C.

1b. Charges distribution

Electric field as a superposition

Let us imagine several charges that are placed in space at known locations as shown in figure 1.4. If another charge q_p is placed at point (x_p, y_p, z_p) then it will experience a total force due to each of these charges combined.

$$\overline{F} = k_e \frac{q_p q_1}{r_1^2}\hat{r}_1 + k_e \frac{q_p q_2}{r_2^2}\hat{r}_2 + \ldots + k_e \frac{q_p q_N}{r_N^2}\hat{r}_N \tag{1.9}$$

The unit vectors $\hat{r}_1$, $\hat{r}_2$, … $\hat{r}_N$ are unit vectors pointing from the charges towards point *p* as in the figure. N is the total number of charges. We can rearrange equation 1.9 as

$$\overline{F} = q_p \sum_{j=1}^{N} k_e \frac{q_j}{r_j^2}\hat{r}_j \tag{1.10}$$

If we compare equation 1.10 to 1.6 and 1.7, one can deduce a similar result.

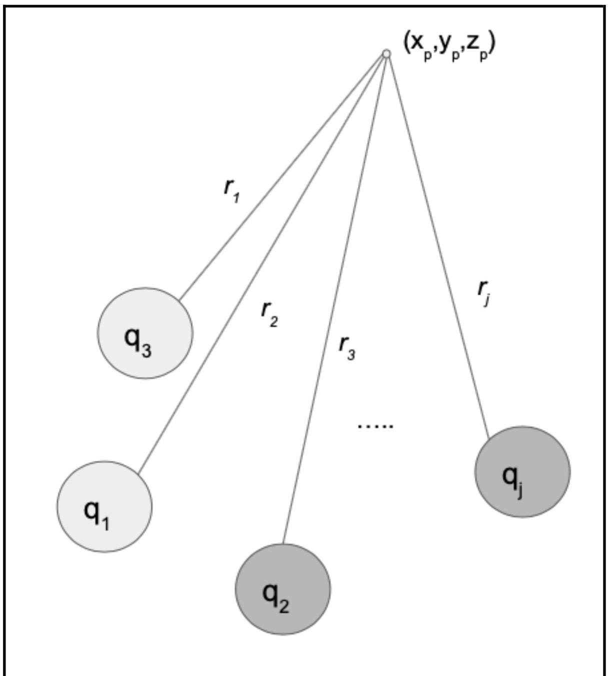

Figure 1.4: Electric field due to many static charges.

$$\overline{F} = q_p \overline{E} \tag{1.11}$$

$$\overline{E} = \sum_{j=1}^{N} k_e \frac{q_j}{r_j^2}\hat{r}_j \tag{1.12}$$

The unit vector $\hat{r}_j = \frac{\overline{r}_j}{r_j}$, which can be written as

$$\hat{r}_j = \frac{(x_j - x_p)\hat{x} + (y_j - y_p)\hat{y} + (z_j - z_p)\hat{z}}{\sqrt{(x_j - x_p)^2 + (y_j - y_p)^2 + (z_j - z_p)^2}} \tag{1.13}$$

If we write the electric field in terms of x,y and z components and apply equation 1.13, we can expand equation 1.12 into three equations

$$E_x = \sum_{j=1}^{N} k_e q_j \frac{x_j - x_p}{\sqrt[3/2]{(x_j - x_p)^2 + (y_j - y_p)^2 + (z_j - z_p)^2}} \tag{1.14a}$$

$$E_y = \sum_{j=1}^{N} k_e q_j \frac{y_j - y_p}{\sqrt[3/2]{(x_j - x_p)^2 + (y_j - y_p)^2 + (z_j - z_p)^2}} \tag{1.14b}$$

$$E_z = \sum_{j=1}^{N} k_e q_j \frac{z_j - z_p}{\sqrt[3/2]{(x_j - x_p)^2 + (y_j - y_p)^2 + (z_j - z_p)^2}} \tag{1.14c}$$

The charges distributed in space generate a total field that is the summation of the electric fields produced by each charge.

Large number of charges in a volume

Now let's look back at figure 1.4 and this time we will increase the number of charges dramatically inside a specific volume *V* as in figure 1.5.

Let's take a very small volume element *dV* centered at a point (x,y,z) inside the total volume *V*. dV is considered to be very small such that the variation of the charges' locations inside it has insignificant effect on the field at point *p* compared to the case when considering that all the charges are located at its center. In this case we can assume that the total charge *q* of *dV* is the summation of all charges inside it. When dealing with large number of charges, it is more reasonable to consider charge density ρ which is charge per unit volume. In this case charge inside *dV* is $q(x, y, z) = \rho(x, y, z)dV$.

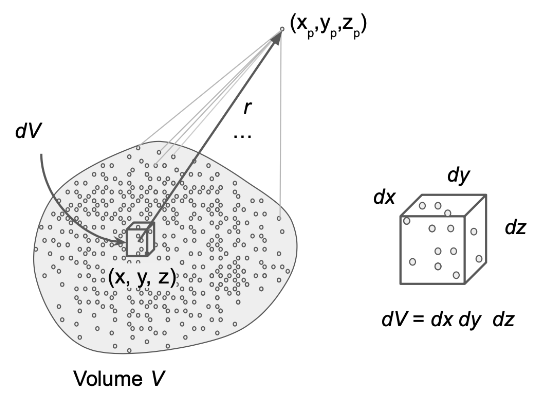

Figure 1.5. Large number of charges distributed in a volume.

Applying this assumption to equations 14 by replacing the summation with integration, q_j by $\rho(x, y, z)dV$ and $\left(x_j,\ y_j,\ z_j\right)$ by $(x,\ y,\ z)$ we obtain

$$E_x = \int_V k_e \rho(x, y, z) \frac{x - x_p}{\sqrt[3/2]{\left(x-x_p\right)^2 + \left(y-y_p\right)^2 + \left(z-z_p\right)^2}} dV \quad (1.15a)$$

$$E_y = \int_V k_e \rho(x, y, z) \frac{y - y_p}{\sqrt[3/2]{\left(x-x_p\right)^2 + \left(y-y_p\right)^2 + \left(z-z_p\right)^2}} dV \quad (1.15b)$$

$$E_z = \int_V k_e \rho(x, y, z) \frac{z - z_p}{\sqrt[3/2]{\left(x-x_p\right)^2 + \left(y-y_p\right)^2 + \left(z-z_p\right)^2}} dV \quad (1.15c)$$

Example 1.1:

Consider charges distributed on a line that is placed on the z-axis as shown in figure 1.6 , write an expression for the electric field produced by these charges. Here, the line carries a constant charge density of ρ_l C/m.

Here the charges are distributed on a line instead of a volume with constant charge density of ρ C/m. Hence, the integration in equations 15 will be performed over a line instead of a volume.

$$E_z = k_e \rho_l \int_{x=-\infty}^{\infty} \frac{z - z_p}{\sqrt[3/2]{\left(x_p\right)^2 + \left(y_p\right)^2 + \left(z-z_p\right)^2}} dz = k_e \rho_l \left[\frac{1}{\sqrt{\left(x_p\right)^2 + \left(y_p\right)^2 + \left(z-z_p\right)^2}} \right]_{z=-\infty}^{z=\infty} = 0 \quad (1.16)$$

From equation 1.16, we find that there is no electric field component along the z axis, which is the direction of the charges. For the x component.

$$E_x = k_e \rho_l \int_{x=-\infty}^{\infty} \frac{(x-x_p)}{\sqrt[3/2]{(x_p)^2+(y_p)^2+(z-z_p)^2}} dz$$

$$= \frac{k_e \rho_l (x-x_p)}{(x-x_p)^2+(y-y_p)^2} \left[\frac{(z-z_p)}{\sqrt{(x-x_p)^2+(y-y_p)^2+(z-z_p)^2}} \right]_{z=-\infty}^{z=\infty}$$

$$= 2k_e \rho_l \frac{-x_p}{(x-x_p)^2+(y-y_p)^2} \qquad (1.17)$$

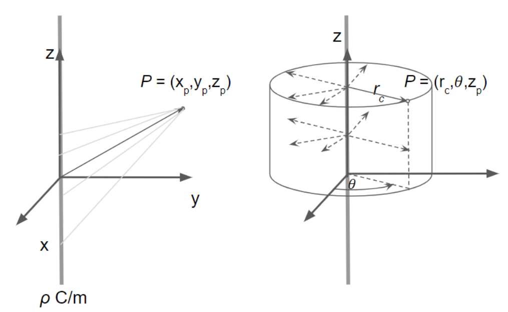

Figure 1.6, Charges distributed on an infinite line along the z axis.

Similarly, the y component is

$$E_y = 2k_e \rho_l \frac{-y_p}{(x-x_p)^2+(y-y_p)^2} \qquad (1.18)$$

The electric field produced by an infinite line of charges is

$$\overline{E} = 2k_e \rho_l \frac{1}{(x-x_p)^2+(y-y_p)^2} \left((x - x_p)\,\widehat{x} + (x - x_p)\,\widehat{y} \right) \qquad (1.19)$$

If we use the cylindrical coordinates in figure 1.6, where $r_c = \sqrt{(x - x_p)^2 + (y - y_p)^2}$, we can write equation 1.19 as

$$\overline{E} = 2k_e \rho_l \frac{1}{r_c^2} \left((x - x_p)\,\widehat{x} + (x - x_p)\,\widehat{y} \right) \qquad (1.20)$$

or

$$\overline{E} = 2k_e \rho_l \frac{1}{r_c} \left(\frac{(x-x_p)}{r_c}\,\widehat{x} + \frac{(x-x_p)}{r_c}\,\widehat{y} \right) = \frac{2k_e \rho_l}{r_c}\,\widehat{r_c} \qquad (1.21)$$

Where $\widehat{r_c}$ is a unit vector pointing at the cylindrical radial direction. So the electric field produced by an infinite line of charges placed on the z-axis is emitted in a cylindrical radial direction and its amplitude is decreasing by the inverse of the distance from the wire.

Example 1.2:

Consider charges distributed on a sheet that is placed in the z-y plane as shown in figure 1.7, write an expression for the electric field produced by these charges. Here, the sheet carries a constant charge density of ρ_s C/m^2.

Here the charges are distributed on a sheet instead of a volume with constant charge density of ρ_s C/m^2. Hence, the integration in equations 1.15 will be performed over an area instead of a volume.

$$E_z = k_e \rho_s \int_{z=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \frac{z-z_p}{\sqrt[3/2]{(x-x_p)^2+(y_p)^2+(z-z_p)^2}} dxdz$$

$$= 0 \qquad (1.22a)$$

$$E_x = k_e \rho_s \int_{z=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \frac{x-x_p}{\sqrt[3/2]{(x-x_p)^2+(y_p)^2+(z-z_p)^2}} dxdz$$

$$= 0 \qquad (1.22b)$$

$$E_y = k_e \rho_s \int_{z=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \frac{-y_p}{\sqrt[3/2]{(x-x_p)^2+(y_p)^2+(z-z_p)^2}} dxdz$$

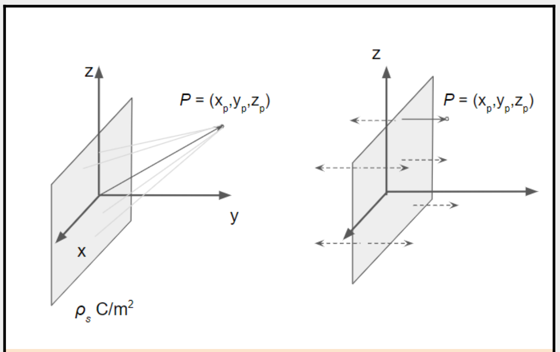

Figure 1.7. Charges distributed on an infinite sheet in the z-x plane.

$$= \int_{x=-\infty}^{\infty} \frac{-2k_e \rho_s y_p}{(x-x_p)^2+(y_p)^2} dx \qquad (1.23a)$$

$$= \int_{x=-\infty}^{\infty} \frac{-2k_e \rho_s}{((x-x_p)/y_p)^2+1} \cdot \frac{1}{y_p} dx \qquad (1.23b)$$

The integration in 1.23 can be solved by change of variables as follows

$$tan\theta = \frac{x-x_p}{-y_p} \rightarrow -y_p \frac{d\theta}{cos^2\theta} = dx \qquad (1.24a)$$

$$x \rightarrow -\infty \Rightarrow \theta \rightarrow -\frac{\pi}{2} \;\; and \;\; x \rightarrow \infty \Rightarrow \theta \rightarrow \frac{\pi}{2} \qquad (1.24b)$$

Using equations 1.24, the integration in 1.23 is simplified to

$$E_y = \int_{\theta=-\pi/2}^{\pi/2} \frac{-2k_e \rho_s}{tan^2\theta + 1} \cdot \frac{1}{y_p} \left(\frac{-y_p}{cos^2\theta} d\theta \right) \qquad (1.25)$$

We know from trigonometry that $1 + tan^2\theta = 1/cos^2\theta$. Hence,equation 1.25 is reduced to

$$\overline{E} = 2\pi k_e \rho_s \hat{y} \qquad (1.26)$$

The electric field produced by an infinite sheet of charges placed in the x-z plane is along the y direction only and its amplitude is constant.

Section review questions

1b-1) Calculate the electric field at a point P which is located at (2 mm, 3 mm, 5 mm) in space near to an infinite line of charges placed on the z-axis with charge density of 10^{-5} C/m. Write the electric field in a vector form.

1b-2) If a charge of amplitude -10^{-15} C is placed at point P in the previous question, what is the amplitude of the force that will be exerted on the charge? If the charge is moved to a new position (4 mm, 6 mm, 5 mm), how much do we need to increase the charge density in the line so to keep the same force as before on the charge?

1b-3) A charge of 2×10^{-15} C is placed at a distance of 300 μm away from an infinite sheet of charges that has a charge density of 0.2 C/m^2. What force does this charge experience and is it a repelling or attraction force? If the charge is moved to a distance of 1 mm, what force does it experience now?

1c. Electric potential

In the previous sections we learn that the presence of a charge or distribution of charges generates an electric field. When another charge is placed in this electric field it experiences a force (pushing or pulling) according to the direction of the electric field and the sign of the charge, $\overline{F} = q\overline{E}$. In order to move this charge along a certain path from one point to another in the presence of this force we need to perform a work on the charge against this force.

Hence, electric potential is defined as the work energy done on a *unit charge* to move it from one point to another.

Potential between two points

If a charge q in figure 1.8 experiences a force $\overline{F}$, then work needed to move it from point P_1 to P_2 on a certain path $\overline{l}$ is the negative of the integration of the force component along the path.

$$W = -\int_{P_1}^{P_2} \overline{F} \cdot \overline{dl} \qquad (1.27)$$

The negative sign is then to stress that the work is done against the applied force. $\overline{dl}$ is a differential vector along the path $\overline{l}$.

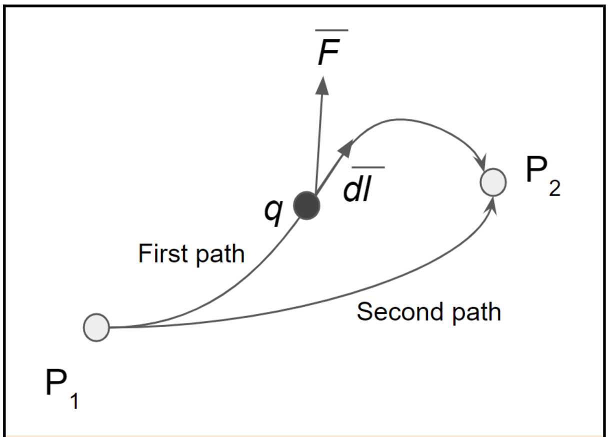

Figure 1.8. The work done to move a charge *q* from point P_1 to P_2.

In terms of electric field field

$$W = -q\int_{P_1}^{P_2} \overline{E} \cdot \overline{dl} \qquad (1.28)$$

When considering the work done per unit charge, then the electrical potential is

$$V_E = \frac{W}{q} = -\int_{P_1}^{P_2} \overline{E} \cdot \overline{dl} \qquad (1.29)$$

From equation 1.29 one can detect that the unit of the electric potential is $J.C^{-1}$ or commonly known as *Volts*. Now, we have learned earlier in the *Physics for engineers I* that the work to move an object from one point to another equals the difference between the potential energies in both positions regardless of the path the object takes in order to perform such transition. The question here is, does this concept hold in the case of the electrical potential? To test this hypothesis, let us examine the case of electrical potential produced by a point charge to move a unit charge between two points as in figure 1.9.

Electric potential due to point charge

Consider the two paths in figure 1.9 between points P_1 and P_2. The first path is $P_1P_1^{'}P_2$ (marked by blue in the figure) and the second path is $P_1P_2^{'}P_2$ (Marked by red in the figure).

Using equation 1.8, we can write the electric potential between the two points as.

$$V_E = -\int_{P_1}^{P_2} \frac{k_e q}{r^2} \hat{r} \cdot \overline{dl} \qquad (1.30)$$

The dot product in equation 1.30 is between the direction of the differential vector along a point on the path and the radial direction. Remember from the vector analysis in the previous course, that this dot product equals zero when the two vectors are orthogonal to each other.

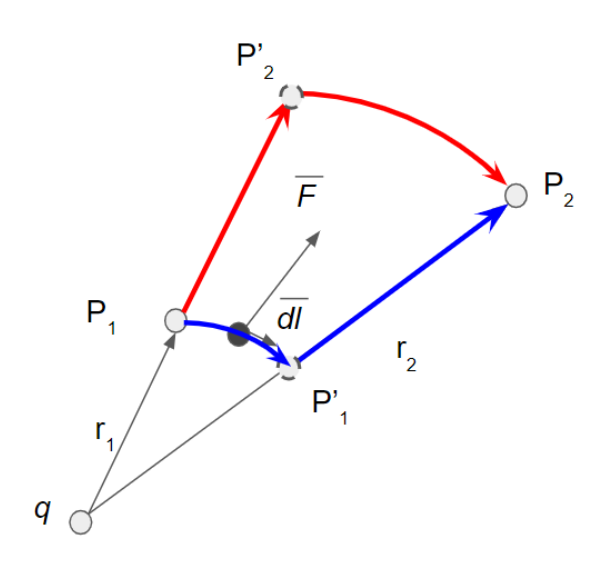

Figure 1.9. Electrical potential by a unit charge between two points P_1 and P_2 when considering two paths.

Let us examine the first path $P_1P_1^{'}P_2$. For that we can break the integration into two parts: one from $P_1 \rightarrow P_1^{'}$ and another from $P_1^{'} \rightarrow P_2$.

$$V_E = -\int_{P_1}^{P_1^{'}} \frac{k_e q}{r^2} \hat{r} \cdot (r_1 d\theta\, \hat{\theta}) - \int_{P_1^{'}}^{P_2} \frac{k_e q}{r^2} \hat{r} \cdot (dr\, \hat{r}) \qquad (1.31)$$

In equation 1.31 the path from $P_1 \rightarrow P_1^{'}$ is along the θ axis. Hence, the dot product $\hat{r} \cdot \hat{\theta} = 0$ and the first term vanishes. In the second term the dot product $\hat{r} \cdot \hat{r} = 1$ and equation 1.31 becomes

$$V_E = -\int_{r_1}^{r_2} \frac{k_e q}{r^2} dr = \left[\frac{k_e q}{r}\right]_{r_1}^{r_2} = k_e q\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad (1.32)$$

Now we consider the second path $P_1P_2^{'}P_2$ which breaks into two parts: one from $P_1 \rightarrow P_2^{'}$ and another from $P_2^{'} \rightarrow P_2$.

$$V_E = -\int_{P_1}^{P_2'} \frac{k_e q}{r^2}\hat{r}\cdot(dr\,\hat{r}) - \int_{P_1'}^{P_2} \frac{k_e q}{r^2}\hat{r}\cdot(r_2 d\theta\,\hat{\theta}) \quad (1.33)$$

Here the path from $P_2' \rightarrow P_2$ is along the θ axis and hence the dot product $\hat{r}\cdot\hat{\theta} = 0$. The second term vanishes and equation 1.33 becomes

$$V_E = -\int_{r_1}^{r_2} \frac{k_e q}{r^2} dr = k_e q\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \quad (1.34)$$

This is exactly the same result obtained from the first path. Hence, the electrical potential does not depend on the path but rather the start and end points.

Section review questions

1c-1) A point charge of amplitude $6\text{x}10^{-14}$ C is placed at the origin. What is the potential measured between two points P_1 and P_2, where P_1 is on a sphere centered at the origin with radius of 1 mm and P_2 is placed on another sphere centered at the origin with a radius of 2 mm?

1c-2) If an infinite sheet of charges with charge density of $3\text{x}10^{-3}$ C/m^2 is placed in the x-z plane, then what would be the measured electric potential between two points P_1 that is the 1 mm away from the sheet and P_2 which is 3 mm away from the sheet?

Hint: Use equation 1.29 with a constant electric field produced by the infinite sheet.

1c-3) Consider an infinite line of charges placed on the z axis. What would be the measured potential between two cylinders of radii 1.5 mm and 2.4 mm centered around the z-axis if the line charge density is set to $5\text{x}10^{-5}$ C/m?