Unit Two:

Electrostatics: Gauss's law

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2a. Electric flux

Flux typically represents an effect that passes or flows through a surface. When talking about electric fields, there is really nothing flowing. The electric field represents the effect of a charge or a distribution of charges in space by which it exerts a force on another charge. The charges that cause the field are commonly called the source charges. In this case defining an electrical flux in a closed surface is not a representation of fields flowing but rather a measure of the electric field through the surface.

Electric flux due to a point charge

To get a better understanding, let us study a case that is easy enough to derive a closed form solution then move to a more general scenario. Consider the field produced by a point charge *q* placed at the origin as we discussed in unit one. Now take a closed surface S that presents a slide in volume between two concentric spheres centered around the origin as shown in figure 2.1

The electric field is calculated as the total field through the surface. That would mean adding all the components of the field that are perpendicular to the surface. Mathematically, we write that as an integration over S of the electric field projection to the normal to the surface at each point.

$$\phi_E = \int_S \overline{E} \cdot \overline{ds} \qquad (2.1)$$

The differential surface vector $\overline{ds}$ is always pointing to the normal to the surface. The surface S in figure 2.1 consists of six surfaces and hence we can break the integration into six integrals

$$\phi_E = \sum_{j=1}^{6} \int_{s_j} \overline{E_j} \cdot \overline{ds_j} \qquad (2.2)$$

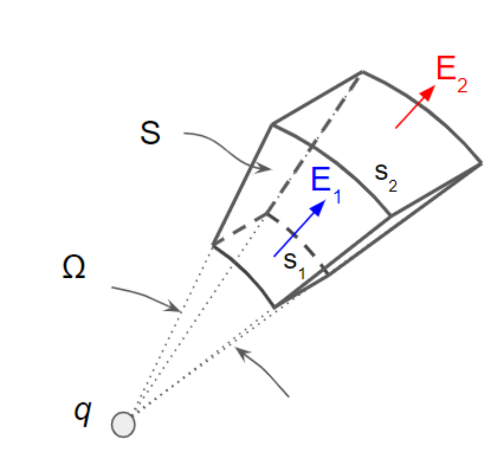


Figure 2.1. Electric flux through a closed surface S duo to a point charge *q*.

As we can notice from the figure that the electric field is always parallel to four surfaces and hence the dot product for these sides is zero. That leaves us with two surfaces: s_1 at the bottom and s_2 at the top.

$$\int_{s_1} \overline{E_1} \cdot \overline{ds_1} + \int_{S} \overline{E_2} \cdot \overline{ds_2} \qquad (2.3)$$

We know from our assumption that the surface S covers a slice in a volume between two concentric spheres (radii r_1 and r_2) that are centered around the origin. Hence the electric field over each surface should be constant. This is because the radius is constant at the surface. For s_1, the electric field equals $\overline{E_1} = k_e \frac{q}{r_1^2}\hat{r}$. Similarly, at s_2 $\overline{E_2} = k_e \frac{q}{r_2^2}\hat{r}$. The vector $\overline{ds_1}$ is pointing at the negative radial direction while $\overline{ds_2}$ points towards the positive r direction. Hence, equation 2.3 is simplified to

$$k_e q\frac{A_1}{r_1^2} - k_e q\frac{A_2}{r_1^2} = k_e q\left(\frac{A_1}{r_1^2} - \frac{A_2}{r_1^2}\right) \quad (2.4)$$

Here A_1 and A_2 are the areas of the surfaces s_1 and s_2 respectively. The ratios $\frac{A_1}{r_1^2}$ and $\frac{A_2}{r_1^2}$ are the solid angles defined by both surfaces. It is clear from figure 2.1 that both surfaces have the same solid angle as the closed surface covers a slice in the volume. Hence, in equation 2.4 the two terms in the parentheses are equal and their subtraction is zero.

$$\phi_E = \int_S \overline{E} \cdot \overline{ds} = 0 \quad (2.5)$$

So the electric flux through the closed surface S due to the point charge q outside the surface is zero.

A point charge inside the surface

Now, what if we study another surface, a sphere for instance that is centered around the point charge. In this case and as illustrated in figure 2.2 the direction of the electric field through the surface is always pointing at the radial direction.

$$\phi_E = \int_S \overline{E} \cdot \overline{ds} = \int_S \frac{k_e q}{r^2}\left(\hat{r} \cdot \hat{r}\right)ds = \frac{k_e q}{r^2}\int_S ds = \frac{k_e q}{r^2}A_S \quad (2.6)$$

In equation 2.6, A_S is the surface area of the sphere, $A_S = 4\pi r^2$.

$$\phi_E = \frac{k_e q}{r^2} \cdot 4\pi r^2 = 4\pi k_e q \quad (2.7)$$

The result in equation 2.7 is quite interesting. The flux through the surface S is constant and does not change regardless how big or small the sphere is selected. It only depends on the charge amplitude *q*.

At this point it might be a good time to return back to the Coulomb's constant that was experimentally determined. This constant happens to be as well

$$k_e = \frac{1}{4\pi\varepsilon_o} \quad (2.8)$$

Now we have a new constant, ε_o, that is defined as ***the vacuum permittivity***. It is also referred to as the vacuum electric constant. ***Vacuum*** typically refers to space that is devoid of matter (or all matters are removed.) When dealing with electric fields, we think of vacuum (or sometimes called **free space**) as a medium at which any radiation travels at the speed of light. To minimize confusion we could set that the vacuum or free space is a reference medium.

Permittivity

Permittivity represents how the medium responds to an applied electric field. If one could think of a medium composed of atoms that has a positive nucleus and negative cloud of electrons. There are two bodies: positive center and negative spherical distribution of charges as shown in figure 2.2a. We assume originally that the two charge bodies have the same center.

If a constant electric field is applied to the atom both nucleus and the electrons cloud will experience a force going in opposite direction

$$\overline{F} = q\overline{E} \quad (2.8)$$

The force will cause the electrons' cloud center to shift from the nucleus. In this case the center of the body of negative charges is not the same as the positive charge. Hence two separate charge centers are present and this medium is referred to as **dielectric** (or two charges).

Electric permittivity is a measure of how the dielectric medium responds to the applied electric field (or how strong the separation of the charges is caused by the field).

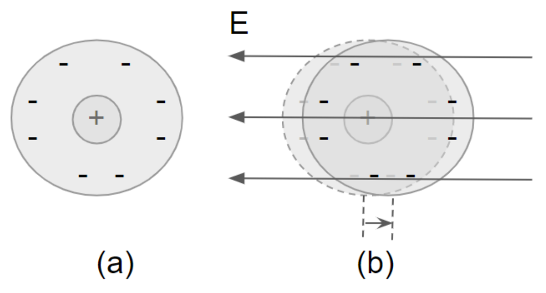


Figure 1.2: (a) Atom composed of positive nucleus and negative electron clouds. (b) The atom responds to the applied electric field by shifting the center of the electron clouds.

This value is measured to a reference, which is the vacuum permittivity. The vacuum permittivity has a value of $\varepsilon_o = 8.8541878128\times10^{-12}\, F\cdot m^{-1}$. Where F stands for Farad. We will cover this unit later when we talk about capacitance. For now, let us take this value as it is. The vacuum permittivity is a reference measure to how dielectric matter responds to an applied electric field. If vacuum or free space refers to absence of matter, hence any medium is expected to have a larger value of permittivity compared to vacuum. The ratio between a medium permittivity, ε, to vacuum is called ***the relative permittivity***, $\varepsilon_r = \frac{\varepsilon}{\varepsilon_o}$.

Flux of point charge

Now that we have briefly introduced the vacuum permittivity and its correlation to Coulomb constant we get back to the electric flux produced by a point charge calculated on a surface of the sphere that surrounds the charge. We can now use the expression of Coulomb's constant (equation 2.8) in equation 2.7.

$$\phi_E = \int_S \overline{E}\cdot\overline{ds} = \frac{4\pi}{4\pi\varepsilon_o}q = \frac{1}{\varepsilon_o}q \quad (2.9)$$

We can rearrange equation 2.9 as

$$q = \varepsilon_o\phi_E = \int_S\left(\varepsilon_o\overline{E}\right)\cdot\overline{ds} = \int_S \overline{D}\cdot\overline{ds} \quad (2.10)$$

Equation 2.10 reads that in order to know the value of the charge inside a surface S, all we need to do is to perform an integration of a new vector $\overline{D} = \varepsilon_o\overline{E}$ through the surface. This new vector quantity is referred to as the **displacement vector** which has units of C/m^2.. We will get back to this later. But we understand that the displacement vector is the multiplication of the electric field by the permittivity. In most of the situations this vector is in the same direction of the electric field. This is not always the case such as the case when the atoms form a lattice (or crystal) form. This is however outside the scope of this course.

Arbitrary surface

The results in the previous section tells us that when a point charge of amplitude q is placed in the center of a sphere, regardless of the sphere radius, the integration of the electric field through the sphere equals the charge divided by the electric permittivity. This concept is actually valid for any closed surface, not necessarily a sphere. The electric flux through a closed surface equals the charge inside the surface divided by the permittivity. Let us keep proof of this generalization as an optional task but we will place it here for those who are interested to understand.

Proof:

1. *Charge outside the surface*

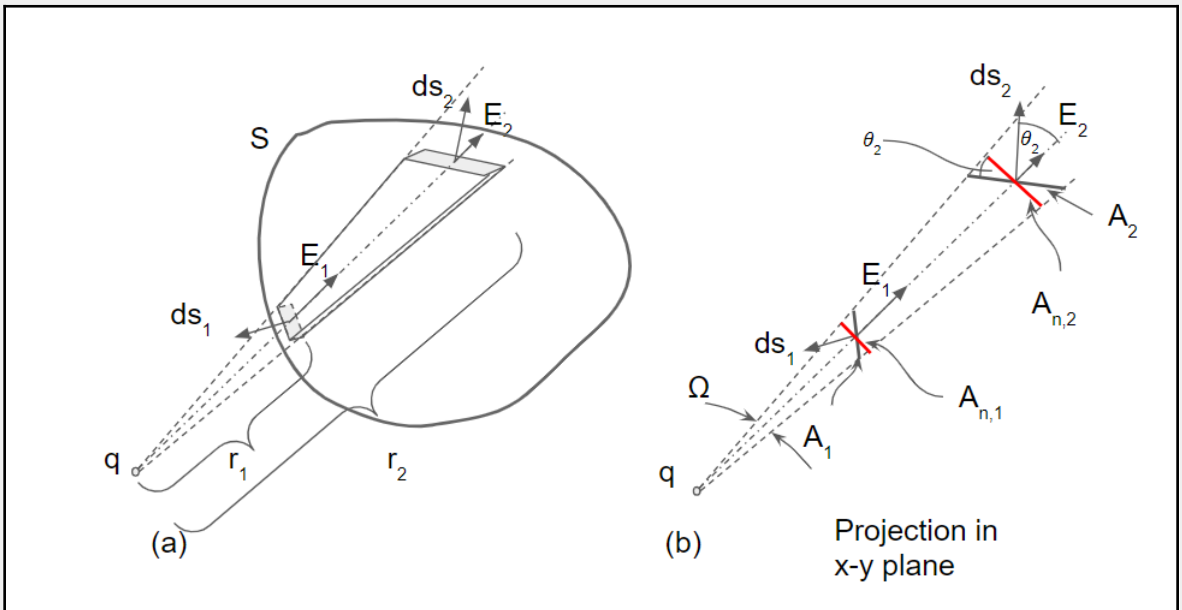


Figure 2.2. (a) The electric flux through a closed surface is represented by breaking the surface into differential slices. (b) The projection of the slice on the x-y plane.

Consider the surface S in figure 2.2a where the charge q is placed outside.Now, we break the surface into small slices as shown in the figure. The slices contain six sides four of which the field is parallel to and hence the dot product in equation 2.1 is zero. The other two surfaces ds_1 and ds_2 are assumed to be small enough that the field is uniform across each one. The flux at ds1 is

$$\phi_{E,1} = \int_{S1} \overline{E} \cdot \overline{ds_1} = \frac{k_e q}{r_1^2} \int_{S1} \hat{r} \cdot \overline{ds_1} = \frac{-k_e q}{r_1^2} \int_{S} ds = \frac{-k_e q}{r_1^2} A_1 cos\theta_1 \quad (2.11)$$

From the projection in figure 2.2b one can see that a surface normal to the electric field would have an area $A_{n.1} = A_1 cos\theta_1$. Then the flux is

$$\phi_{E.1} = -\ k_e q \frac{A_{1n}}{r_1^2} = -\ k_e q\ \Omega \qquad (2.12)$$

Where Ω is the solid angle as explained earlier. Similarly, one can use the same approach to show that for the flux through the surface ds_2 is

$$\phi_{E.2} = k_e q \frac{A_{2n}}{r_2^2} = k_e q\ \Omega \qquad (2.13)$$

The total flux is then $\phi_E = \phi_{E.1} + \phi_{E.2} = -\ k_e q\ \Omega + k_e q\ \Omega\ = 0.$

2. *Charge inside the surface*

We can use a similar approach to show that for a charge inside a surface S has the same flux of that of a spherical surface S2 that is centered around the charge and is contained inside S as shown in figure 2.3.

In the figure to the right, the surface S is broken into differential slices in the same way we did in the first case. However, here we compare the flux through the slice form S to that from a spherical surface S_2 centered around q and is included inside S.

If we take the slice in figure 2.3, front the analysis in equations 2.11 and 2.12 we know that both differential surfaces the one on S and the one S2 sustain the same solid angle Ω. Hence the flux through both of these surfaces are the same. The total flux through the two surfaces should as well be equal.

The total electric flux in equation 2.9 is general through any surface.

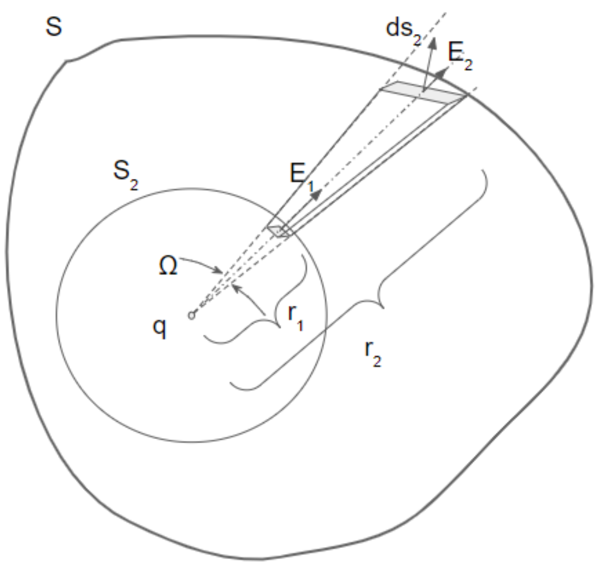


Figure 2.3. Flux by a point charge inside a surface S in equivalence to a spherical surface containing the same charge.

Flux due to multiple charges

If multiple charges present inside the surface S then what would be the electrical flux through the surface? To answer this question we return to equation 2.1. The flux is the integration of the electric field through the surface. Hence, we could say that the flux due to multiple charges is the integration of the total electric field produced by the charges through the surface. If N charges exist inside the surface each produces a field of E_j where j varies from 1 to N, then

$$\phi_E = \int_S \sum_{j=1}^{N} \overline{E}_j \cdot \overline{ds} = \sum_{j=1}^{N} \int_S \overline{E}_j \cdot \overline{ds} = \sum_{j=1}^{N} \phi_{E.j} \quad (2.14)$$

Where $\phi_{E,j}$ is the flux through the surface S due to a point charge q_j inside the surface. We know from before that the flux duo to a point charge inside a surface S is $\phi_{E.j} = \frac{1}{\varepsilon_o} q_j$. Then the total flux due to all the charges is

$$\phi_E = \frac{1}{\varepsilon_o} \sum_{j=1}^{N} q_j = Q/\varepsilon_o \quad (2.15)$$

Where Q is the total charge inside the volume surrounded by the surface S, or $Q = \sum_{j=1}^{N} q_j$. If we have large number of charges in the volume that is incubated by the surface S, then the total charge is $Q = \int_V \rho(r) dV$ and in this case we can write Gauss's law for electric flux as

$$\phi_E = \int_S \overline{E} \cdot \overline{ds} = \frac{1}{\varepsilon_o} \int_V \rho(r) dV \quad (2.16)$$

Electric field and flux by a sphere of charges

The expression in equation 2.16 states that the flux over a closed surface S equals the integration of the electric field over a closed surface and it is also equal to the integration of the charge density over a volume of charges divided by ε_o. So, let us consider the following example of a sphere of charges that has a uniform charge density (ρ is constant) as depicted in figure 2.4. The question is how to calculate the electric field at a spherical surface S due to charge distribution in a spherical volume V?

One straightforward way is to simply perform the integrations in equation 2.16. We know that the charge density is constant, hence the integration on the right hand side of equation 2.16 is simplified to

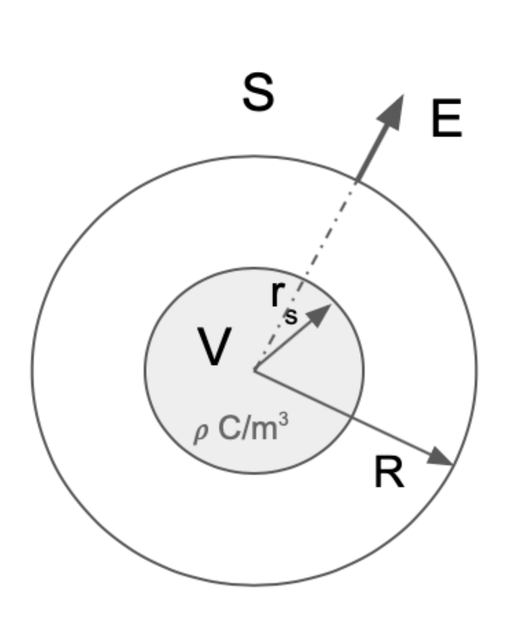


Figure 2.4. A sphere of charges of volume V inside a closed concentric spherical surface S.

$$\phi_E = \frac{\rho}{\varepsilon_o} \int_V dV = \frac{4\pi\rho}{3\varepsilon_o} r_s^3 = Q/\varepsilon_o \quad (2.17)$$

Where the total charge of the sphere is $Q = \frac{4\pi\rho}{3} r_s^3$. It is time now to tackle the first integration in equation 2.16. The main issue here is what is the field due to the sphere at a distance R from the center. To solve this issue let us logically examine the simplified model in figure 2.5. Here we assume distinguished changes in the volume. As seen at any point P on the surface S, one can select a plane *L* that contains the point and the center of the sphere around which there will always exist a symmetrical distribution of the charges around that plane. In figure 2.5, this plane is represented by the dashed-dotted line.

If we draw the electric field vectors due to the charges inside the volume, one can logically predict that the total electric field should be in the radial direction due to the symmetry of the charges around the plane. We also can use the very same logic to conclude that the amplitude of the electric field will always be constant through the surface S. In this way,

$$\phi_E = \int_S \overline{E} \cdot \overline{ds} = E\int_S (\hat{r} \cdot \hat{r})ds$$
$$= 4\pi R^2 E \qquad (2.18)$$

Substituting these results in equation 2.16

$$4\pi R^2 E = Q/\varepsilon_o \rightarrow \overline{E} = \frac{Q}{4\pi\varepsilon_o R^2}\hat{r} \qquad (2.19)$$

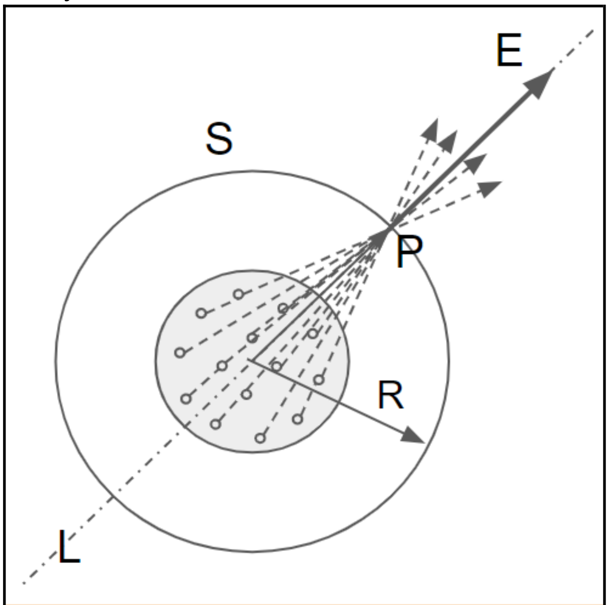


Figure 2.5. Electric field due to sphere of charges.

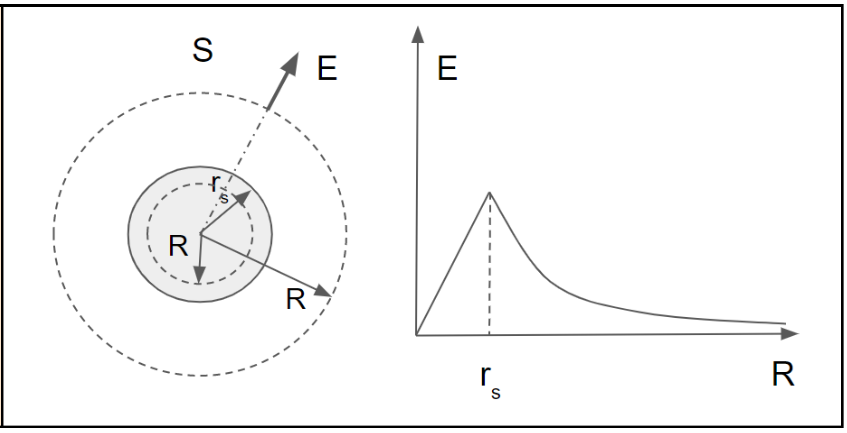
This is exactly the same electric field produced by a point charge of amplitude Q placed at the center of the sphere.

What if the surface S is placed inside the volume V, or $R<r_s$? In this case the total charge $Q = \frac{4\pi\rho}{3} R^3$ and the electric field is

$$\overline{E} = \frac{Q}{4\pi\varepsilon_o R^2}\hat{r} = \frac{4\pi\rho}{3} R^3 \times \frac{1}{4\pi\varepsilon_o R^2}\hat{r} = \frac{\rho R}{3\varepsilon_o}\hat{r} \qquad (2.20)$$

In this case the electric field inside the volume V is increasing linearly with the radius till a maximum value of $E_{max} = \rho r_s/3\varepsilon_o$. Once the surface S becomes larger than the sphere of charges, the electric field starts to reduce as an inverse to radius square.

Figure 2.6. The electric field amplitude as a function of the surface S radius R. The electric field inside the sphere increases linearly with R. Outside the sphere it decreases with an inverse to the square of the radius.



Section review questions

2a-1) A closed surface S contains a total amount of chargers of 2 C, what would be the flux on a spherical surface S that contains those charges? What if we change the surface to be a cylinder that contains the charges? What would the electric flux be if the surface does not include the charge?

2a-2) A point charge is placed in the origin of the coordinate system. If the electric field measured at a distance 2 mm from the charge is $3x10^{-5}$ V/m, what is the electric flux through a cylinder of 3 mm radius that contains the same charge?

2a-3) an infinite sheet of charges is placed in the x-y plane. The sheet produces an electric field of 1 V/m.

(a) What is the charge density on the sheet?
(b) What is the electric flux through the surfaces S1 and S2 as shown in the figure.

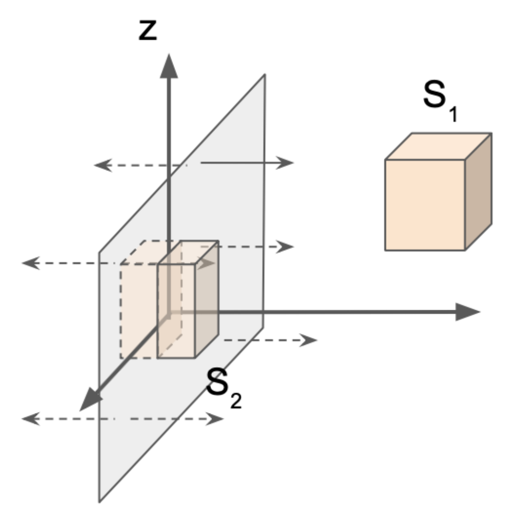


Figure 2.7. Electric flux through two surfaces S1 and S2. One surface is outside the sheet and the other contains part of the sheet. The surfaces are cube shaped with side length of 2 mm.

2a-4) Calculate the electric field and flux produced by a sphere of charges that has a charge density of $3x10^{-3}$ C/m^3 and it has a radius of 3 mm on a spherical surface that includes the sphere and it has a radius of 5 mm.

2b. Electric field lines

Electric field lines by point charge

We know that the electric field emitted by a point charge of amplitude q is $\overline{E} = \frac{q}{4\pi\varepsilon_o r^2}\hat{r}$. The amplitude of the field is decreasing as the inverse square of the distance from the charge, *r*. The direction of the field is along the radial direction $\hat{r}$. So, if we draw the electric field direction as arrows and the surfaces of constant amplitude as dashed lines we construct the graph in figure 2.8.

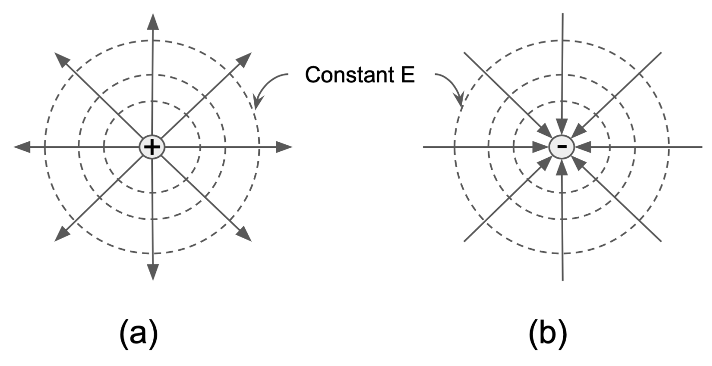


Figure 2.8. Electric field lines (arrows) and surfaces of equal electric field amplitude or equipotential surfaces in dashed lines for (a) positive charge where lines are pointing outwards and (b) negative charge where the electric lines are pointing inwards.

For the case of positive charge the value of q is positive and hence the electric field lines are in the $\hat{r}$ direction pointing outwards from the charge. In the case of a negative charge, the value of q is negative and hence the electric field lines are in the $-\hat{r}$ direction pointing towards the charge. The surfaces of equal electric field are referred to as the equipotential surface. According to equation 1.32 in unit one the potential needed to bring a charge from infinity to any point on this surface is $V_E = k_e q/R$ and it is always constant on the surface of a sphere of radius R. One thing we can easily notice in the plots in figure 2.8 is that the electric field lines are always perpendicular to the equipotential surface.

Electric field lines by two point charges

Consider two point charges q1 and q2 placed at a distance apart from each other. Without lack of generality, we could assume that the two charges are placed on the x-axis. How to represent the electric field lines and the equipotential surfaces?

First let us take the two charges in figure 2.8 and bring them near to each other on the x-axis where q_1 is placed at $x =- L/2$ and q_2 is places at $x = L/2$. At any point in space $P \equiv (x, y, z)$ the electric field is the superposition of the electric fields generated by each charge.

$$\overline{E} = \frac{q_1}{4\pi\varepsilon_o r_1^2}\hat{r}_1 + \frac{q_2}{4\pi\varepsilon_o r_2^2}\hat{r}_2 \qquad (2.22)$$

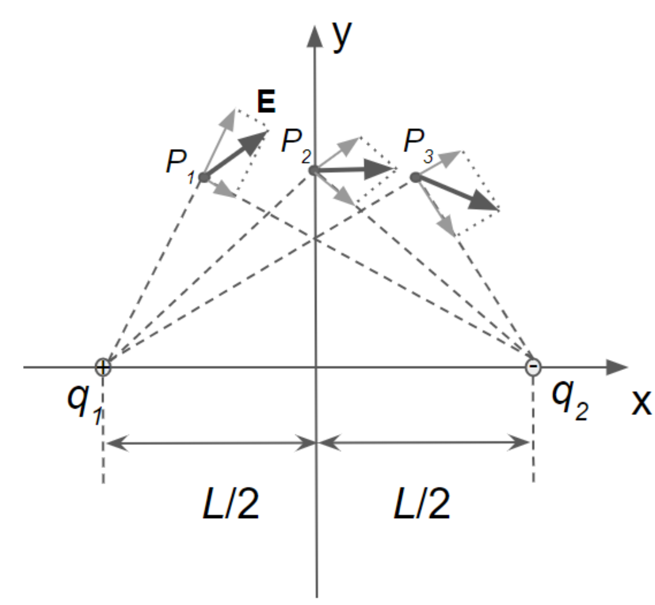


Figure 2.9. Field due to two point charges placed on the x-axis.

As shown in figure 2.9, the charge q_1 is positive and hence the field lines are pointing outwards while q_2 is negative and the lines are pointing towards the charge. At point P_1 that is closer to charge q_1, the amplitude of the electric field due to q1 is stronger than that due to q_2 (remember that the electric field is inversely proportional to the square of the distance from the charge.) The summation of the two electric fields then yields a vector that has a positive y component. At a point P2 in the middle plane (one the y-axis for instance) the distance from the two charges are equal and hence both fields have the same magnitude. The superposition results in a vector that points in the positive x direction. Finally for a point P3 that is closer to q2, the total electric field has a negative y-component as shown in figure 2.9.

The electric field lines and equipotential surfaces are depicted in figure 2.10. The electric field lines start from the positive charge and end on the negative charge. For equal charges, the potential is zero on a plane that is in the middle between the two charges. To the left or the middle place the potential is positive while it becomes negative to the right (closer to the negative charge.)

The distribution in figure 2.10 would change if the charges are not equal. If for instance the positive charge amplitude becomes double of the negative charge, then the plots in figure 2.11a are produced.

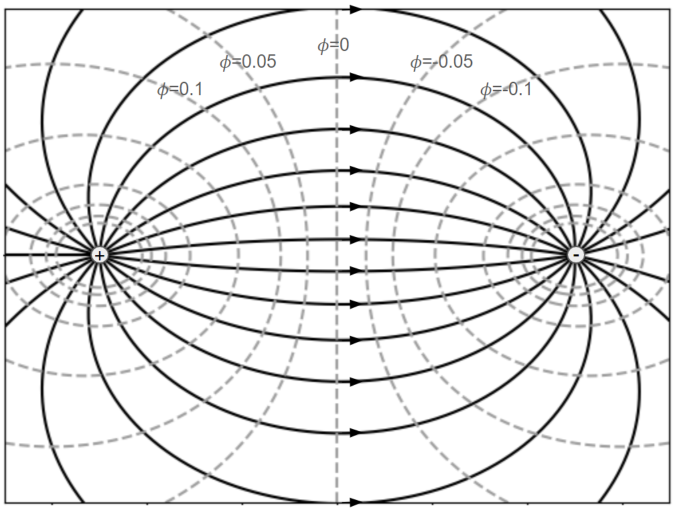


Figure 2.10. Electric field lines (solid) and equipotential surfaces (dashed lines) for two equal charges separated by a distance L.

The plots show that the zero equipotential surface is pushed further away from the center towards the smaller charge (the negative charge in our case). This surface is pushed even further when the positive charge amplitude increases to four times that of the negative charge as shown in figure 2.11b.

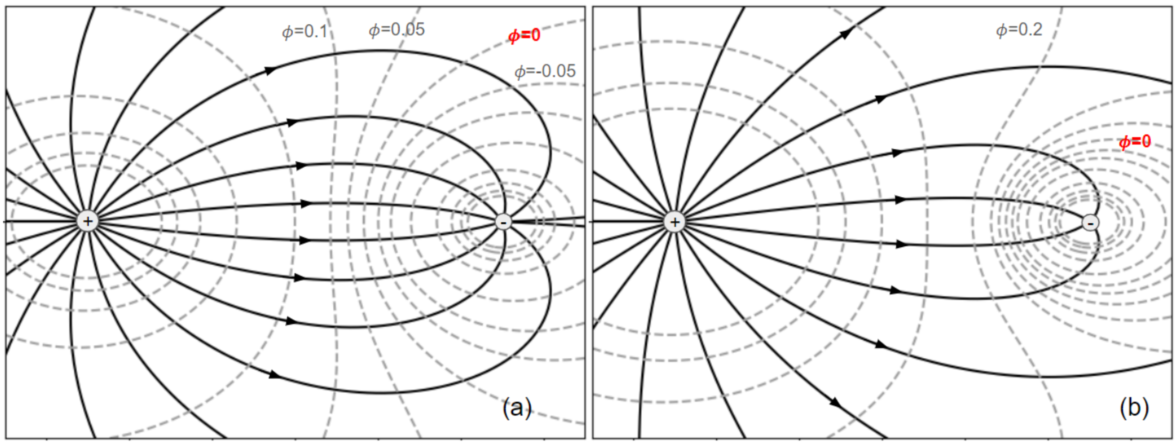


Figure 2.11. (a) Electric field lines (solid) and equipotential surfaces (dashed) for a positive charge that is double of the negative one. (b) When the positive charge is four times the negative one.

Electric field lines by two infinite sheet of charges

Let us consider now the case when two infinite sheets of charges are brought close to each other where one has a positive charge density ρ_s and the other has a negative charge density $-\rho_s$. The electric field produced by each sheet is constant and it has amplitudes

$$E_1 = 2\pi k_e \rho_s = \frac{\rho_s}{2\varepsilon_o} \text{ and } E_2 = -\frac{\rho_s}{2\varepsilon_o} \tag{2.23}$$

The field E_1 lines are pointing away from the sheet while those of E_2 are pointing towards the sheet as illustrated in figure 2.12.

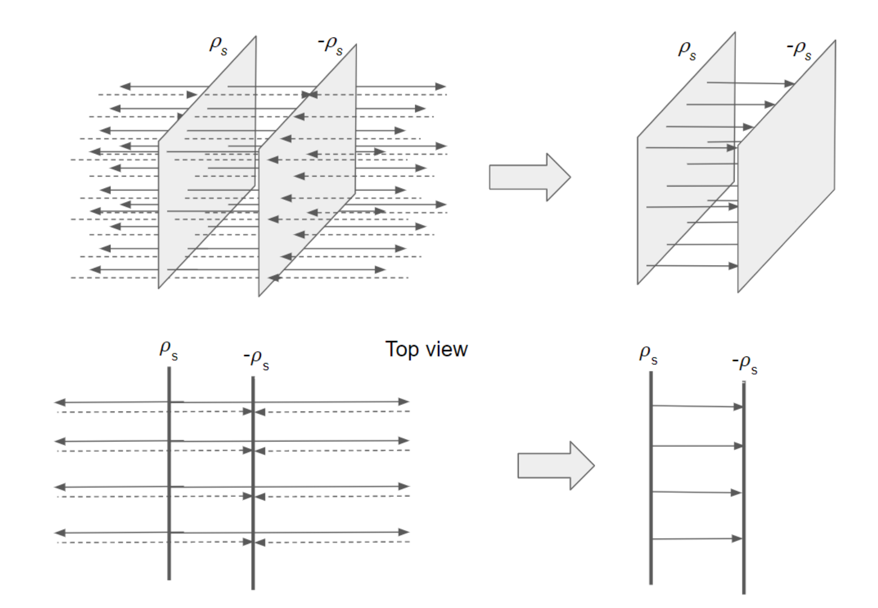


Figure 2.13. The electric field lines as a superposition of the two electric fields produced by two infinite sheets of opposite and equal charge densities. The electric field is constant between the sheets and zero outside. The lines are parallel and start from the positive to the negative charges.

.In the figure there are three regions of interest: to the left of the positive sheet, in between the sheets and to the right of the sheet. To the left of the positive sheet E1 is pointing towards the negative x direction. Hence, $\overline{E_1} = -\frac{\rho_s}{2\varepsilon_o}\hat{x}$. For the negative sheet the electric field is pointing towards the sheet in the positive x direction and hence $\overline{E_2} = \frac{\rho_s}{2\varepsilon_o}\hat{x}$. The total electric in this region is then $\overline{E} = \overline{E_1} + \overline{E_2} = 0$. Similar argument can be stated for the third region to the right of the negative charge where the fields have opposite signs and the net total is zero. Between the two sheets, the two electric fields are pointing in the positive x direction and hence the total field is

$$\overline{E} = \frac{\rho_s}{2\varepsilon_o}\hat{x} + \frac{\rho_s}{2\varepsilon_o}\hat{x} = \frac{\rho_s}{\varepsilon_o}\hat{x} \tag{2.23}$$

The electric field lines between the two infinite sheets are parallel and start from the positive sheet and end at the negative one. The electric field between the sheet iis constant and has an amplitude of $\frac{\rho_s}{\varepsilon_o}$ V/m. The electric field vanishes outside the sheets.

Section review questions

2b-1) Draw the electric field line produced by one point charge (one positive and one negative) and illustrate the equipotential surfaces on the graph.

2b-2) Draw roughly the electric field lines between two point charges of equal amplitude and opposite sign and roughly illustrate the equipotential surfaces.

2b-3) Draw the electric field lines by two infinite sheets of charges placed 3 mm apart and have equal surface charge density of 5x10-3 C/m^2. What is the amplitude of the electric field?

2c. Electric field and conductors

A conductor is a material that has an excess amount of free electrons that can move freely inside but not leaving the surface. This is the case with metals where there are many free electrons such that when an electric field is applied a large number of them will mov. Now, within the limits of electrostatics, charges are considered to be static or not moving. Hence, the electrons will arrange themselves to a situation where they cease motion. When motion stops then the total force on the electrons inside the metal should be balanced to zero. That means that the internal electric field vanishes or

$$E_{inside} = \int_V \frac{\rho(V)}{4\pi\varepsilon_o r^2} dV = 0. \qquad (2.24)$$

Hence the volume charge density, ρ, should be zero inside the4 metal for the integration to vanish for any volume V. If this is the case, then how can charged metal form in electrostatic limits? If the volume charge density is zero then for any charge to exist on the metallic body it has to be on the surface in a form of a surface charge density σ.

Conducting sphere

If we draw the electric field vectors due to the charges on the surface of a charged conducting sphere, then one can logically predict that the total electric field should be in the radial direction due to the symmetry of the charges. We can also use the very same logic to conclude that the amplitude of the electric field will always be constant through the surface S. In this way, the electric flux is

$$\phi_E = \int_S \overline{E} \cdot \overline{ds} = E \int_S (\hat{r} \cdot \hat{r}) ds$$

$$= 4\pi R^2 E \qquad (2.24)$$

If the surface S is reduced to be just outside the metallic sphere, then the radius $R \rightarrow r_s$ and $\phi_E \rightarrow 4\pi r_s^2 E$. We know from equation 2.17 that the flux through a surface S is $\phi_E = Q/\varepsilon_o$, where Q is the total charge inside the surface.

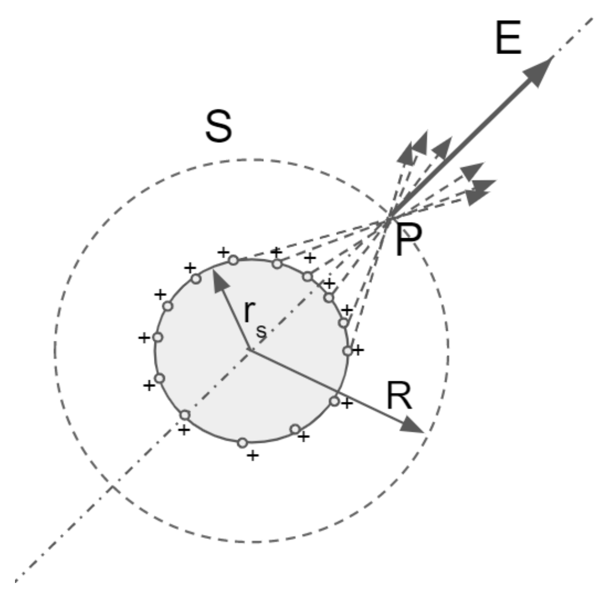


Figure 2.14. Electric field due to charged conductor sphere.

We know that for a conductor the charges only exist on the surface and hence the total charge $Q = 4\pi r_s^2 \sigma$.

$$\phi_E = 4\pi R^2 E = Q/\varepsilon_o \rightarrow \overline{E} = \sigma/\varepsilon_o \hat{r} \qquad (2.25)$$

The electric field produced by the metallic sphere on the surface is σ/ε_o pointing normal to the surface (here it is the radial direction.) The question here is would this relation hold for arbitrary conductor shape?

For the conductor in figure 2.15 we consider a small cylindrical surface S around a differential part of the volume. The electric field that contributes to the flux will be from the two surfaces not from the sides as the electric field is normal to the differential surface ds. We know as well that the field inside the conductor is zero. Hence, the flux is only due to the field component outside the conductor.

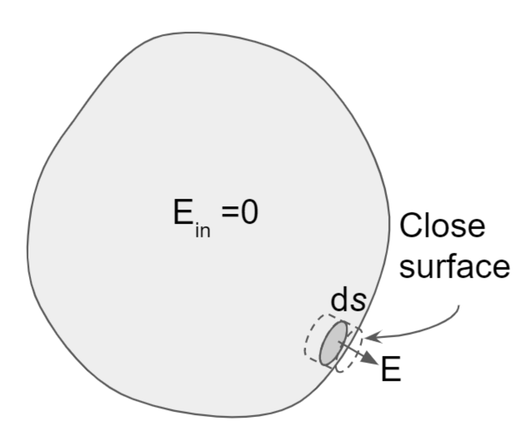


Figure 2.15. Electric field due to arbitrary charged conductor .

The flux $\Delta\phi_E$ through the differential surface ds is $\Delta\phi_E = \overline{E}.\overline{ds} = Q/\varepsilon_o$. The electric field is normal to the surface and hence it is in the same direction as ds. The total charge on the surface ds is $Q = \sigma ds$ where σ is the local surface charge density, Hence the electric field amplitude can be deduced as

$$Eds = Q/\varepsilon_o = \sigma ds/\varepsilon_o \rightarrow E = \sigma/\varepsilon_o \quad (2.26)$$

This is the same result we obtained for a charged metallic sphere.

Charged conductor sheet

Let us consider a metallic sheet as in figure 2.14. If an energy source is applied on it to move the charges to the surface then an electric field is produced outside the sheet. If we select a closed surface S around a differential area ds then as we did before, we find our that the electric field amplitude is σ/ε_o. This is twice the value we found earlier for the electric field produced by an infinite sheet of charges.One can argue that for the infinite sheet the field was emitted in both directions. However, in metal the field could only be present outside. So, we could say that for the case of the conductor with the same surface charge density as a sheet of charges, there exists another field component which compensates for the one that points inside the metal as illustrated in figure 2.16. Hence, the electric field outside is double of that of the sheet of charges.

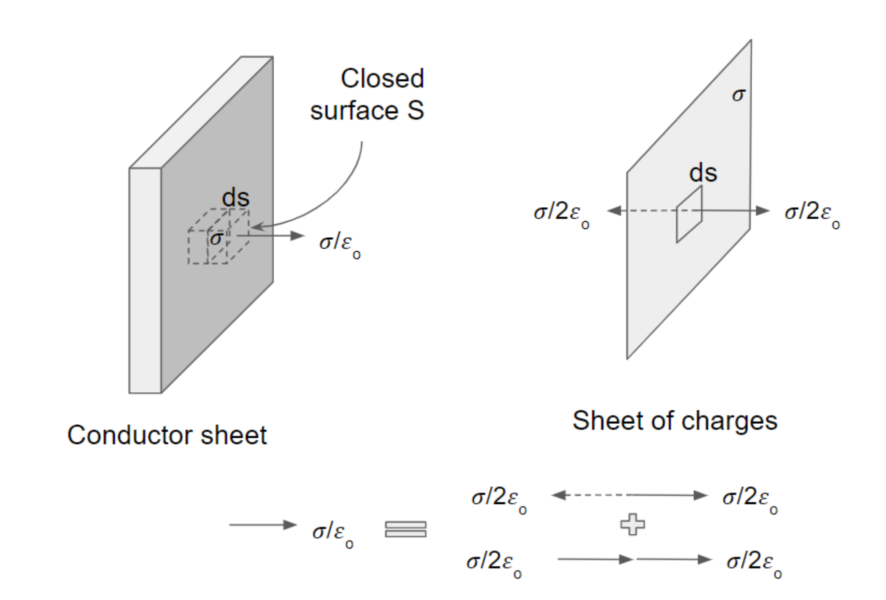


Figure 2.16. Electric field produced by a charged conductor sheet compared to that produced by an infinite sheet of charges.

An extra field is assumed to be present for the conductor to compensate for the electric field component that points inside the metal.

Sor the electric field at any point on the surface of any conductor has an amplitude of σ/ε_o and it is pointing in a direction normal to the surface.

Capacitance

When an energy source is applied on a conductor, under the electrostatic limits, a total charge of $Q = \int_S \sigma\, ds$ is built on the surface, S. The presence of the charges produces an electric field normal to the surface with amplitude σ/ε_o. That results in a potential difference, V_E, between the conductor surface and a reference surface. The amount of charge that is needed to produce 1 V of potential difference is defined as the capacitance *C*.

$$C = Q/V_E \quad (2.27)$$

The unit of the capacitance is Coulomb per volt or Farad. Here 1 Farad represents a capacitance for a system that stores 1 Coulomb of charge and produces an electric potential difference of 1 V.

Self capacitance

When considering an isolated conductor with closed surface S, then the reference surface can be at infinity. In this case the electric potential becomes the work needed to move a unit charge from infinity to the conductor surface. The capacitance in this situation is known as the self capacitance. One simple conducting geometry that we could think of is the conducting sphere. The electric potential is then

$$V_E = \int_{r_s}^{\infty} \overline{E} \cdot \hat{r}\, dr = \int_{r_s}^{\infty} \frac{Q}{4\pi\varepsilon_o r^2} dr = \frac{Q}{4\pi\varepsilon_o r_s} \tag{2.28}$$

The total charge on the sphere is $Q = 4\pi r_s^2 \sigma$ when having a constant surface charge density. The self capacitance in this case is

$$C = Q/V_E = \frac{4\pi r_s^2 \sigma}{r_s \sigma/\varepsilon_o} = 4\pi r_s \varepsilon_o \tag{2.29}$$

Mutual capacitance

Let us consider two conducting surfaces that are charged with opposite charge signs. The electric fields in this case will be leaving the positive surface in normal direction and end up on the negative surface in normal directions. In this case the capacitance of the system of two conductors is the amount of charge stored to produce a potential difference of 1 V between the surfaces. Picking simple geometry allows us to derive closed form expressions for the capacitance.

The electric field at the surface of each conductor has the same amplitude of σ/ε_o and are pointing in the x axis direction out of the positive sheet ending at the negative sheet. We also know from the infinite sheet of charges that the electric field is constant in the space between the two conductors and has the same amplitude. This is ofcourse if we neglect the effect of the finite area A assuming it to be much larger than the spacing L. In this case the potential difference between the two sheets is

$$V_E = \int_{x_1}^{x_2} \overline{E} \cdot \hat{x}\, dx = \frac{\sigma}{\varepsilon_o} \int_{x_1}^{x_2} dx = \frac{\sigma}{\varepsilon_o}(x_2 - x_1)$$
$$= \frac{\sigma}{\varepsilon_o} L \tag{2.30}$$

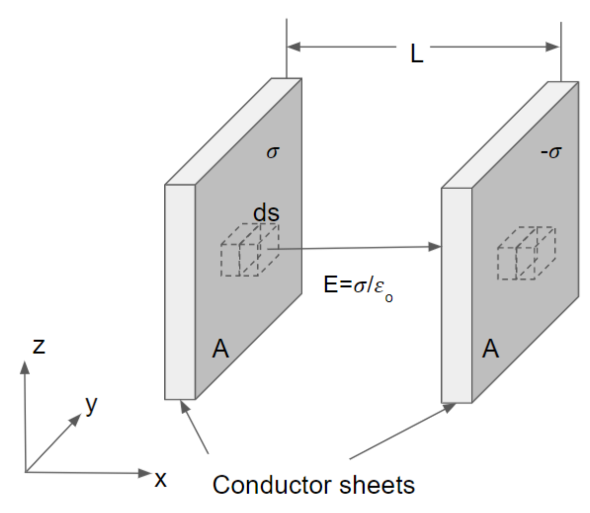


Figure 2.17. Mutual capacitance between two conducting sheets.

The mutual capacitance is then

$$C = Q/V_E = \frac{\sigma A}{\sigma L/\varepsilon_o} = \frac{A\varepsilon_o}{L} \tag{2.31}$$

Let us now consider the case of capacitance between two concentric spheres as shown in figure 2.18.

The electrical potential difference between the two surfaces is $V_E = V_{E,1} - V_{E,2}$. The potential on each surface can be estimated directly from equation 2.28 as $V_{E,1} = \frac{Q}{4\pi\varepsilon_o r_1}$ and $V_{E,2} = \frac{Q}{4\pi\varepsilon_o r_2}$. Notice here the potential $V_{E,2}$ is positive though the charge on the second surface is -Q. This is due to the fact that the dot product between the electric field and the normal to the surface is negative. This compensates for the negative charge sign. The potential difference is then

$$V_E = \frac{Q}{4\pi\varepsilon_o r_1} - \frac{Q}{4\pi\varepsilon_o r_2} = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad (2.32)$$

The mutual capacitance is then

$$C = Q/V_E = 4\pi\varepsilon_o/\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad (2.33)$$

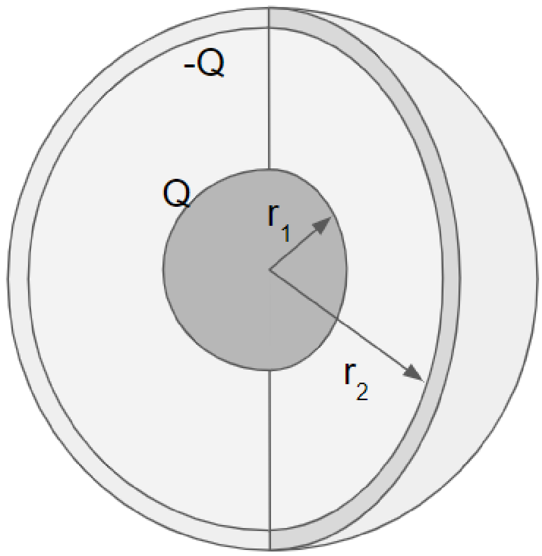


Figure 2.18. Mutual capacitance between two concentric spheres.

Finally, let us examine another simple geometry of two concentric cylinders as in figure 2.19. Here, we will again assume that the length of the cylinder is much larger than the spacing between the two surfaces and hence we can neglect the finite extension of the conductor.

We know that the electric field on the surface of the conductor is constant and it equals σ/ε_o and a direction normal to the surface. If the total charge on the inner cylinder is Q, then the field at the inner surface is $\overline{E}_1 = \frac{Q}{2\pi r_1 h \varepsilon_o}\hat{r}$. Similarly the electric field on the second surface is $\overline{E}_2 = \frac{-Q}{2\pi r_2 h \varepsilon_o}(-\hat{r}) = \frac{Q}{2\pi r_2 h \varepsilon_o}\hat{r}$. Hence, one can say that the electric field at any location r is $\overline{E} = \frac{Q}{2\pi r \varepsilon_o}\hat{r}$. The potential difference between the two surfaces is then

$$V_E = \int_{r_1}^{r_2} \frac{Q}{2\pi r h \varepsilon_o} dr = \frac{Q}{2\pi\varepsilon_o h} ln\left(\frac{r_2}{r_1}\right) \quad (2.34)$$

The mutual capacitance is then

$$C = Q/V_E = 2\pi\varepsilon_o h/ln\left(\frac{r_2}{r_1}\right) \quad (2.35)$$

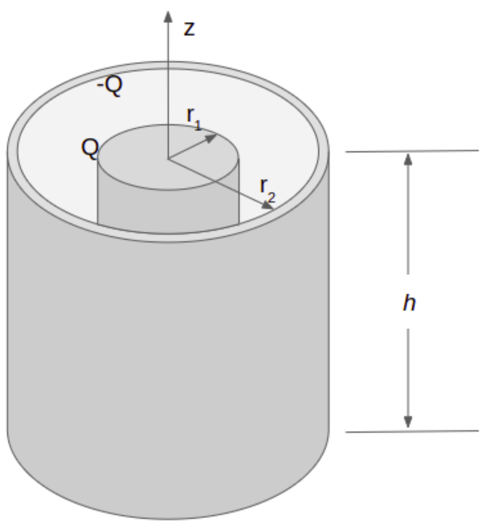


Figure 2.19. Mutual capacitance of two concentric cylinders.

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Type	Geometry	Capacitance
Self	sphere	$4\pi r_s \varepsilon_o$
Mutual	Two parallel sheets	$A\varepsilon_o / L$
Mutual	Concentric cylinders	$2\pi\, h\, \varepsilon_o / ln\left(\frac{r_2}{r_1}\right)$
Mutual	Concentric spheres	$4\pi\varepsilon_o / \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

T**able 2.1.** Summary of capacitance for simple geometries.

Section review questions

2c-1) Calculate the electric field at the surface of the following conductors assuming a charge of 2 C is uniformly distributed.
a) Metallic sphere of 3 mm radius.
b) Metallic cylinder of 1 mm radius and 2 cm height.
c) Metallic cube of 2 mm side.

2c-2) What is the radius of an isolated metallic sphere if a self capacitance of 10 μF is measured?

2c-3) A coaxial cable consists of two concentric metallic cylinders. What is the inner cylinder radius if a capacitance of 50 μF is measured? The outer cylinder radius is 5 mm.

2c-4) Calculate the capacitance of the following geometries
a) An isolated metallic sphere of 6 mm radius.
b) Two concentric spheres of radii 2 mm and 7 mm.
c) Two concentric cylinders of radii 1 mm and 4 mm.
d) Two parallel sheets of an area of 4 mm^2 and a separation of 1 mm.