SCHOOL OF
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Unit Three:
More on Electric field

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3a. Electric field in differential form

Electric field as gradient of the potential

The potential difference between two points is

$$V_E = \phi_2 - \phi_1 = -\int_{p1}^{p2} \overline{E} \cdot \overline{dl} \qquad (3.1)$$

The potential difference between two points P1 and P2 is a line integration over a trajectory *l*(x,y) as illustrated in figure 3.1. Let us now break this trajectory into multiple small steps Δl.

The potential difference between P2 and P1 can be written as a sum of the potential differences between the intermediate points along the trajectory as follows

$$\phi_2 - \phi_1 = \left(\phi_2 - \phi_{(n)}\right) + \left(\phi_{(n)} - \phi_{(n-1)}\right) + \dots + \left(\phi_{(2)} - \phi_{(1)}\right) + \left(\phi_{(2)} - \phi_1\right) \qquad (3.2)$$

If we assume the coordinates of two consecutive points on the trajectory to be $P_{(j)} = (x_j, y_j)$ and $P_{(j+1)} = (x_j + \Delta x_j, y_j + \Delta y_j)$, then we can write the potential differential between them as is

$$\phi_{(j+1)} - \phi_{(j)} = \phi(x_j + \Delta x_j, y_j + \Delta y_j) - \phi(x_j, y_j) \qquad (3.3)$$

If the steps are too small, we can expand the expression above using Taylor expansion keeping only the first order term and neglecting higher power as

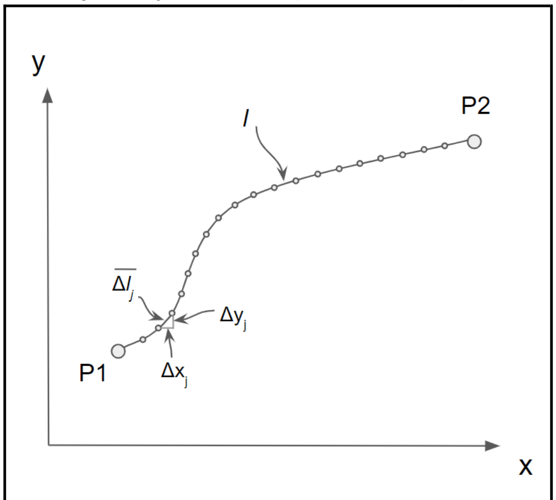


Figure 3.1. Breaking the trajectory *l* into smaller steps and evaluating the potential differences along the trajectory.

$$\phi(x_j + \Delta x_j, y_j + \Delta y_j) \approx \phi(x_j, y_j) + \Delta x_j \frac{\partial \phi}{\partial x}\Big|_{(xj,yj)} + \Delta y_j \frac{\partial \phi}{\partial y}\Big|_{(xj,yj)} + O(\Delta x_j^2, \Delta y_j^2) \qquad (3.4)$$

Using this approximation the expression in 3.3 is simplified to

$$\phi_{(j+1)} - \phi_{(j)} \approx \Delta x_j \frac{\partial \phi}{\partial x}\Big|_{(xj,yj)} + \Delta y_j \frac{\partial \phi}{\partial y}\Big|_{(xj,yj)} \qquad (3.5)$$

From the illustration in figure 3.1, we can say that $\overline{\Delta l}_j = \left(\Delta x_j, \Delta y_j\right)$. Hence, equation 3.5 can be written as a dot product of two vectors

$$\phi_{(j+1)} - \phi_{(j)} \approx \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)_{(xj,yj)} \cdot \left(\Delta x_j, \Delta y_j\right) = \overline{\nabla}\phi_j \cdot \overline{\Delta l}_j \qquad (3.6)$$

Where the **nabla operator** in the two dimensional case is $\overline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. The potential difference between points P1 and P2 as in equation 3.2 is the summation of the potential differences between each two consecutive points. Hence, we can write that

$$\phi_2 - \phi_1 = \sum_{j=0}^{N+1} \left(\phi_{(j+1)} - \phi_{(j)}\right) \qquad (3.7)$$

Notice that in the notation in equation 3.7, we set $\phi_{(0)} = \phi_1$ and $\phi_{(N+1)} = \phi_2$, where N is the total number of intermediate points between P1 and P2. Using the approximation in 3.6,

$$\phi_2 - \phi_1 \approx \sum_{j=0}^{N+1} \overline{\nabla}\phi_j \cdot \overline{\Delta l}_j \tag{3.8}$$

If the steps size becomes very small, $\Delta l \rightarrow 0$,the summation in equation 3.8 becomes integration.

$$V_E = \phi_2 - \phi_1 \approx \int_l \overline{\nabla}\phi \ \cdot \overline{dl} \tag{3.9}$$

If we compare equations 3.9 with 3.1, we can say that the electric field at any point of space equals the **gradient** of the electric potential difference

$$\overline{E} =- \overline{\nabla}\phi \tag{3.10}$$

Flux and electric field divergence

From the discussion in unit two, we know that the electric flux through a closed surface S equals the total charge inside the surface divided by the free space permittivity

$$\psi_E = \int_S \overline{E} \cdot \overline{ds} = Q/\varepsilon_o \tag{3.11}$$

Let us consider an infinitesimal cubical volume $\Delta V = \Delta x \Delta y \Delta z$ in space that is enclosed in surface ΔS as shown in figure 3.2. The volume has a uniform charge density of ρ C/m^3.

The total flux is a result of adding six components, one from each surface. The electric field vector is $\overline{E} = \left(E_x, E_y, E_z\right)$. For the first surface S1, the component $\overline{ds} =- dydz\,\hat{x}$ as it is pointing to the negative x direction. The flux through the surface is $\psi_1 = \int_S \overline{E_1} \cdot (dydz\,\hat{x})$. If we assume the surface is small that field is constant, then

$$\psi_1 = -\ E_x \int_S dydz =- E_x \Delta y \Delta z \tag{3.12}$$

For the second surface, the electric field is $E_{x.2} = E_x(x + \Delta x)$. If the distance Δx is very small, then we can apply the appreciation to Taylor expansion as

$$E_{x.2} = E_x(x + \Delta x) \approx E_x + \Delta x \frac{\partial E_x}{\partial x} \tag{3.13}$$

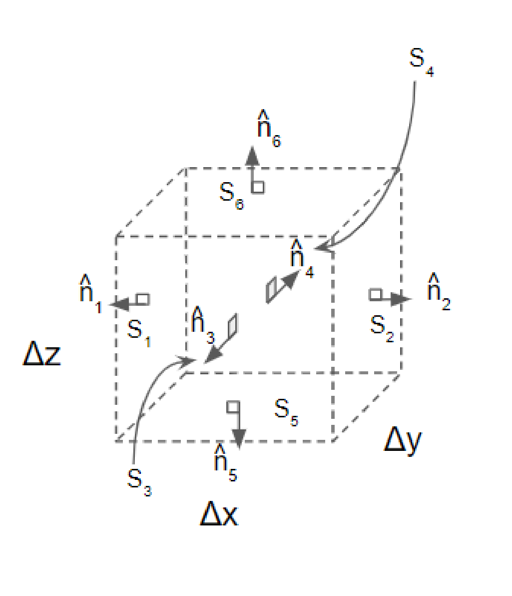


Figure 3.2. Electric flux through an infinitesimal cubic surface.

The normal component here is The electric flux from the second surface is then

$$\psi_2 = \left(E_x + \Delta x \frac{\partial E_x}{\partial x}\right) \Delta y \Delta z \tag{3.14}$$

The flux by the two surfaces S1 and S2 is then.

$$\psi_{12} = \psi_1 + \psi_2 = - E_x \Delta y \Delta z + \left(E_x + \Delta x \frac{\partial E_x}{\partial x}\right)\Delta y \Delta z = \Delta x \Delta y \Delta z \frac{\partial E_x}{\partial x} \quad (3.15)$$

In a very similar way, we can derive an approximate expression for the flux by surfaces S3 and S4 as

$$\psi_{34} = \psi_3 + \psi_4 = - E_y \Delta x \Delta z + \left(E_y + \Delta y \frac{\partial E_y}{\partial y}\right)\Delta x \Delta z = \Delta x \Delta y \Delta z \frac{\partial E_y}{\partial y} \quad (3.16)$$

Also, for the surfaces S5 and S6,

$$\psi_{56} = \psi_5 + \psi_6 = - E_z \Delta x \Delta y + \left(E_z + \Delta z \frac{\partial E_z}{\partial z}\right)\Delta x \Delta y = \Delta x \Delta y \Delta z \frac{\partial E_z}{\partial z} \quad (3.17)$$

The total flux is estimated by adding the results from equations 3.15 to 3.17.

$$\psi_E = \psi_{12} + \psi_{34} + \psi_{56} = \Delta x \Delta y \Delta z \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \quad (3.18)$$

The term in parentheses in equation 3.18 is known the **divergence** of the electric field and we typically write it as $\overline{\nabla}.\overline{E}$, where the **nabla operator** is $\overline{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$. Let us now substitute the expression in 3.18 into 3.11.

$$\psi_E = \Delta x \Delta y \Delta z (\overline{\nabla}.\overline{E}) = Q/\varepsilon_o \quad (3.19)$$

We can write the total charge Q in terms of volume charge density as $Q = \rho \Delta x \Delta y \Delta z$. Then equation 3.19 can be reduced to

$$\overline{\nabla}.\overline{E} = \rho/\varepsilon_o \quad (3.20)$$

Zero potential in a closed trajectory

We know that the electric potential is the work needed to move a unit charge from one point to another. If the unit charge is moved on a closed loop trajectory such that it ends at the starting point then the work is zero and correspondingly the produced electric potential difference is zero. Mathematically, we write this as follows.

$$V_E = \phi_1 - \phi_1 = - \oint_l \overline{E}.\overline{dl} = 0 \quad (3.21)$$

Where l is the closed trajectory. For simplicity we can set this trajectory as an infinitesimal square path in the x-y plane as demonstrated in figure 3.3.

Let us assume coordinate of the four points as

$P0 = (x, y)$

$P1 = (x, y + \Delta y)$

$P2 = (x + \Delta x, y + \Delta y)$

$P3 = (x + \Delta x, y)$

For the path from point Po to P1, the trajectory l is along the y axis. For a very small separation the integration can be represented by the area under the linear curve in figure 3.4. The shaded region consists of a rectangle and triangle. The area of this region is

$$\phi_1 - \phi_0 = - \Delta y\left[E_y(x, y) - \frac{1}{2}\left(E_y(x, y + \Delta y) - E_y(x, y)\right)\right]$$
$$= - \frac{\Delta y}{2}\left(E_y(x, y + \Delta y) + E_y(x, y)\right) \quad (3.22)$$

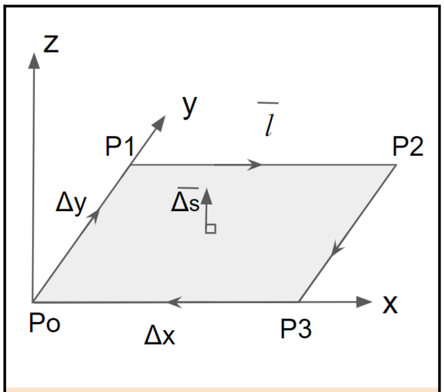


Figure 3.3. Zero electric potential for a unit charge to move on a closed loop.

Using Taylor expansion and assuming Δy is very small, we can approximate the expression in 3.22 as

$$\phi_1 - \phi_0 = \phi_{01} = -\frac{\Delta y}{2}\left(E_y(x,y) + \Delta y\frac{\partial E_y}{\partial y} + E_y(x,y)\right) = -\Delta y\, E_y(x,y) - \frac{\Delta y^2}{2}\frac{\partial E_y}{\partial y} \quad (3.23)$$

Similarly from P1 to P2, the trajectory is along the x axis and the integration can be performed as a sum of two areas, a rectangle and triangle

$$\phi_{12} = -\frac{\Delta x}{2}\left(E_y(x+\Delta x, y+\Delta y) + E_y(x, y+\Delta y)\right) \quad (3.24)$$

Using the expansion and approximating for small Δx and Δy

$$\phi_{12} = -\frac{\Delta x}{2}\left(E_x(x,y) + \Delta x\frac{\partial E_x}{\partial x} + \Delta y\frac{\partial E_x}{\partial y} + E_x(x,y) + \Delta y\frac{\partial E_x}{\partial y}\right)$$
$$= -\Delta x E_x(x,y) - \Delta x\Delta y\frac{\partial E_x}{\partial y} - \frac{\Delta x^2}{2}\frac{\partial E_x}{\partial x} \quad (3.25)$$

From P2 to P3

$$\phi_{23} = \frac{\Delta y}{2}\left(E_y(x+\Delta x, y+\Delta y) + E_y(x+\Delta x, y)\right)$$
$$= \Delta y E_y(x,y) + \Delta x\Delta y\frac{\partial E_y}{\partial x} + \frac{\Delta y^2}{2}\frac{\partial E_y}{\partial y} \quad (3.26)$$

Notice that the sign here is positive due to the fact that the path from P2 to P3 is along the negative y direction. Finally, from P3 to P4

$$\phi_{34} = \frac{\Delta x}{2}\left(E_x(x+\Delta x, y) + E_y(x,y)\right) = \Delta x E_x(x,y) + \frac{\Delta x^2}{2}\frac{\partial E_x}{\partial x} \quad (3.26)$$

Adding the three potential differences

$$V_E = \phi_{01} + \phi_{12} + \phi_{23} + \phi_{34} = \Delta y\Delta x\frac{\partial E}{\partial y} - \Delta x\Delta y\frac{\partial E}{\partial x} = \Delta x\Delta y\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \quad (3.27)$$

The expression in equation 3.27 can be considered as a dot product of the normal to surface bounded by the closed trajectory, $\overline{\Delta S} = \Delta x\Delta y\,\hat{z}$ and another vector that has z component that is equal to $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$. To identify this vector, let us recall the definition of the **Curl operator** of the vector $\overline{E}$.

$$\overline{\nabla} \times \overline{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)\hat{z} \quad (3.28)$$

Comparing with the z component, one can say that the electric potential in equation 3.27 is

$$V_E = \Delta x\Delta y\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -(\overline{\nabla} \times \overline{E}) \cdot \overline{\Delta S} \quad (3.29)$$

In integral form we know that $\overline{\Delta S} = \int \overline{ds}$. If we assume the curl of the electric field is constant inside the area ΔS then

$$V_E = -\int_S (\overline{\nabla} \times \overline{E}) \cdot \overline{dS} = -\oint_l \overline{E}.\overline{dl} = 0 \quad (3.30)$$

From equation 3.30, we can reach two results. The first one is that the integration over a closed loop of the electric field equals the integration of the curl of the field over the area covered by the trajectory.

$$\oint_l \overline{E}.\overline{dl} = \int_S (\overline{\nabla} \times \overline{E}) \cdot \overline{dS} \quad (3.31)$$

The second result is that for the electrostatic limit, the curl of the electric field equals zero.

$$\overline{\nabla} \times \overline{E} = 0 \quad (3.32)$$

We can summarize the three relations in the following table

No	Relation	Reads as	Meaning
1	$\overline{E} = -\overline{\nabla}\phi$	Nabla Phi	Electric field is proportional to the gradient of the electric potential
2	$\overline{\nabla}.\overline{E} = \rho/\varepsilon_o$	Div E	The divergence of the electric field is proportional to the volume charge density
3	$\overline{\nabla} \times \overline{E} = 0$	Curl E	The curl of the electric field is zero in the electrostatics limits.

Table 3.1. Differential relations for the electric field in electrostatics limits.

2b. Electric dipole

Two point charges

Let us revisit the two point charges example we studied earlier for the electric field lines. For two opposite charges that are separated by a distance *d*, the electric potential at any point P is the summation of the electric field potential caused by each charge.

The electric potential at point P is

$$\phi_P = \frac{q}{4\pi\varepsilon_o r^+} - \frac{q}{4\pi\varepsilon_o r^-} \quad (3.33)$$

where

$r^+ = \sqrt{x^2 + y^2 + (z - d/2)^2}$ and

$r^- = \sqrt{x^2 + y^2 + (z + d/2)^2}$

Expanding the square term inside the square root

$$r^+ = \sqrt{x^2 + y^2 + z^2 - zd + d^2/4} \quad (3.34)$$

We know that $r = \sqrt{x^2 + y^2 + z^2}$. We use this relation and taking r outside the square root we obtain

$$r^+ = r\sqrt{1 - \frac{zd}{r^2} + \frac{d^2}{4r^2}} \quad (3.35)$$

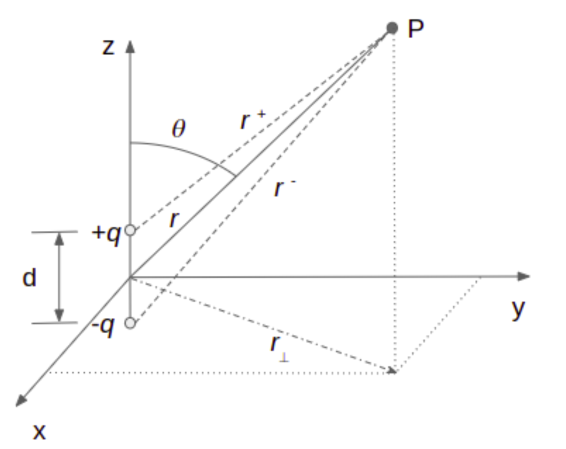


Figure 3.4. Electric dipole formed by two point charges separated by a distance d.

If the point P is far away from the dipole such that d<<r, then one can cancel the last term in equation 3.35 and use Taylor expansion to approximate the distance from the positive charge to the point P as $r^{+} \approx r\sqrt{1 - \frac{zd}{r^2}}$. Similarly we can write $r^{-} \approx r\sqrt{1 + \frac{zd}{r^2}}$. Using this approximation, we can express $1/r^{+}$ and $1/r^{-}$ using Taylor expansion as

$$\frac{1}{r^{+}} \approx \frac{1}{r}\left(1 + \frac{zd}{2r^2}\right) \quad (3.36a)$$

$$\frac{1}{r^{-}} \approx \frac{1}{r}\left(1 - \frac{zd}{2r^2}\right) \quad (3.36b)$$

Using the approximation in equations 3.63 into the potential in equation 3.33 we obtain

$$\phi_P = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r^{+}} - \frac{1}{r^{-}}\right) = \frac{q}{4\pi\varepsilon_o r}\left(\left(1 + \frac{zd}{2r^2}\right) - \left(1 - \frac{zd}{2r^2}\right)\right) = \frac{z}{4\pi\varepsilon_o r^3} \cdot qd \quad (3.37)$$

In equation 3.37 the potential is proportional to a quantity qd that is the product of the charges amplitude and the separation between them. This quantity is known as the **dipole momentum** and we typically use the letter *p* for it.

$$\phi_P = \frac{z}{4\pi\varepsilon_o r^3} \cdot p \quad (3.38)$$

From the geometry in figure 3.4, we find that $z = r cos\theta$. Hence, $z/r = cos\theta$. We can express the potential in terms of the angle θ as

$$\phi_P = \frac{cos\theta}{4\pi\varepsilon_o r^2} \cdot p \quad (3.39)$$

The electric field of the dipole can be obtained from the gradient of the potential $\overline{E} = \overline{\nabla}\phi$.

$$E_z = -\frac{\partial}{\partial z}\left(\frac{z}{4\pi\varepsilon_o r^3} \cdot p\right) = -\frac{p}{4\pi\varepsilon_o} \cdot \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) \quad (3.40)$$

We know that $z = r cos\theta$, then

$$E_z = \frac{p}{4\pi\varepsilon_o r^3} \cdot \left(3cos^2\theta - 1\right) \quad (3.41)$$

For the x and y components we obtain the following expressions

$$E_x = \frac{3p}{4\pi\varepsilon_o r^5} \cdot xz \quad (3.42a)$$

$$E_y = \frac{3p}{4\pi\varepsilon_o r^5} \cdot yz \quad (3.42a)$$

As depicted in figure 3.4, the dipole axis is on the z-direction that is normal to the z - y plane. Hence, we can represent the electric field in two component one parallel to the dipole axis $E_{||} = E_z$ and one normal to the dipole axis $E_{\perp} = \sqrt{E_x^2 + E_y^2}$.

$$E_{\perp} = \frac{3pz}{4\pi\varepsilon_o r^5} \cdot \sqrt{x^2 + y^2} \quad (3.43)$$

Observing the geometry in figure 3.4 one can deduce that $r_{\perp} = \sqrt{x^2 + y^2} = r \cdot sin\theta$. Using this relation equation 3.43 can be simplified to

$$E_{\perp} = \frac{3p\, sin\theta\, cos\theta}{4\pi\varepsilon_o r^3}\,. \quad (3.43)$$

Figure 3.5 shows the electric field lines generated by the electric dipole.

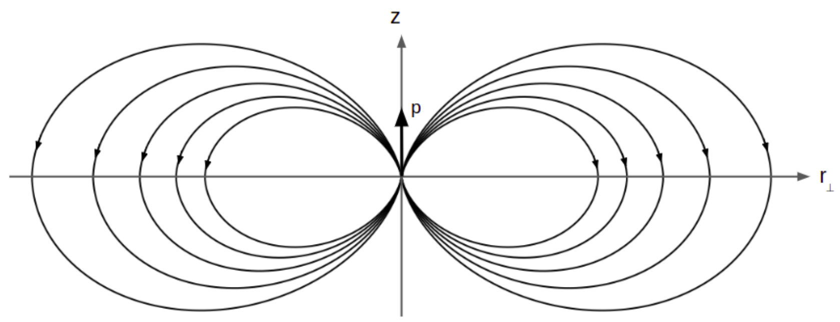


Figure 3.5. Electric field lines for an electric dipole that is formed by two charges of equal amplitudes and opposite signs and it is placed along the z axis.

As seen in the figure, the normal electric field component ,$E_{\perp}$, vanishes for θ=0°,90° and -90°. At these angles the electric field lines are parallel to the dipole.

A collection of positive and negative charges

In the previous section we represented two point charges of equal amplitude and opposite sign as one dipole that has a potential that depends on the charges' amplitude multiplied by the distance between them. Now, let us consider a collection of different charges, positive and negative, distributed in a volume V as shown in figure 3.6.

The potential at point P due to a charge qi is $\phi_{Pi} = \frac{q_i}{4\pi\varepsilon_o r_{pi}}$. The total electric potential at point P due to all the charges is the summation of that caused by each individual charge, or

$$\phi_P = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_o r_{pi}} \quad (3.44)$$

Where N is the total number of charges inside the volume V. The distance r_{pi} is

$$r_{pi}^2 = (x - x_i)^2 + (y - x_i)^2 + (z - z_i)^2$$
$$= x^2 + y^2 + y^2 + x_i^2 + y_i^2 + z_i^2$$
$$- 2(xx_i + yy_i + zz_i) \quad (3.45)$$

Recalling that $\bar{r} = (x, y, z)$ and $\bar{r}_i = (x_i, y_I, z_I)$. The equation 2.45 can be written as

$$r_{pi}^2 = r^2 + r_i^2 - 2\bar{r}\cdot\bar{r}_i \quad (3.46)$$

where $\bar{r}\cdot\bar{r}_i = xx_i + yy_i + zz_i$.

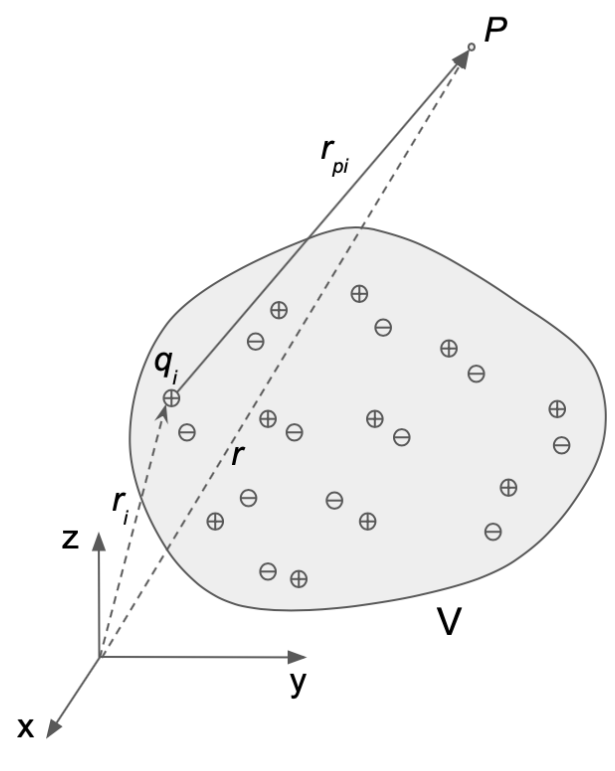


Figure 3.6. Field due to two point charges placed on the x-axis.

If the point P is moved far away from the volume V such that $r \gg r_i$, then we can write the distance r_{pi} as

$$r_{pi} \approx \sqrt{r^2 - 2\bar{r} \cdot \bar{r}_i} = r\sqrt{1 - 2\left(\bar{r} \cdot \bar{r}_i\right)/r^2} \quad (3.47)$$

The term $1/r_{pi}$ can then be approximated as

$$1/r_{pi} = \frac{1}{r}\frac{1}{\sqrt{1-2\left(\bar{r}\cdot\bar{r}_i\right)/r^2}} \approx \frac{1}{r}\left(1 + \left(\bar{r} \cdot \bar{r}_i\right)/r^2\right) \quad (3.48)$$

In equation 3.48, we applied Taylor expansion and applied the approximation that $\left(\bar{r} \cdot \bar{r}_i\right)/r^2 \ll 1$ neglecting the higher power terms. Using this approximation, equation 3.44 can be rewritten as

$$\phi_P \approx \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_o r}\left(1 + \left(\bar{r} \cdot \bar{r}_i\right)/r^2\right) \quad (3.49)$$

Rearranging the terms

$$\phi_P \approx \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_o r} + \sum_{i=1}^{N} \frac{q_i \bar{r}_i \cdot \hat{e}_r}{4\pi\varepsilon_o r^2} \quad (3.50)$$

Here we define a unit vector $\hat{e}_r = \bar{r}/r$ that is along the direction from the origin of the coordinate system to the point P. Moving the common factors in equation 3.50 outside the summation we obtain the following

$$\phi_P \approx \frac{1}{4\pi\varepsilon_o r}\sum_{i=1}^{N} q_i + \frac{1}{4\pi\varepsilon_o r^2}\sum_{i=1}^{N} q_i \bar{r}_i \cdot \hat{e}_r \quad (3.51)$$

The first term in equation 3.50 depends only on the total charge inside the volume $\sum_{i=1}^{N} q_i$ while the second term depends on the charges and their distribution $\sum_{i=1}^{N} q_i \bar{r}_i \cdot \hat{e}_r$. If we assume equal amount of positive and negative charges, then the total charge inside the volume should equal zero, $\sum_{i=1}^{N} q_i = 0$. Hence, the first term in equation 3.51 vanishes.

$$\phi_P = \frac{1}{4\pi\varepsilon_o r^2}\left(\sum_{i=1}^{N} q_i \bar{r}_i\right) \cdot \hat{e}_r \quad (3.52)$$

If we define $\bar{p} = \sum_{i=1}^{N} q_i \bar{r}_i$ as the total dipole momentum of the total charges inside the volume V, then equation 3.52 is reduced to

$$\phi_P = \frac{\bar{p} \cdot \hat{e}_r}{4\pi\varepsilon_o r^2} \quad (3.53)$$

This is a very similar result as equation 3.38 when using the dot product representation in terms of vector projections as $\bar{p} \cdot \hat{e}_r = p cos\theta$. Hence, when observing in the far field, the electric potential produced by charges distribution inside the volume is equivalent to that produced of a single dipole of momentum $\bar{p}$ that depends on the charges' distribution inside the volume.

3c. Polarization vector and dielectric medium

Recall the dielectric medium we introduced in the first unit. There we talked about a medium that is made of molecules (or single atoms as in figure 3.7). When an electric field is applied on the medium, it exerts a force on the charges. Positive and negative charges experience opposite forces and hence the center of charges will shift as illustrated in figure 3.7. If the medium occupies a volume V as in the figure, then we can estimate the electric potential at point P due to the medium as we did with a collection of charges in equation 3.53. For a molecule that is located at location $\overline{r}_i = (x_i, y_i, z_i)$, the center of negative and positive charges will be shifted when the field is applied. If the electric field is along the z axis, then the location of the charge centers can be related as

$$\overline{r_i^+} = \overline{r_i^-} + \delta\hat{z} \qquad (3.54)$$

Here, δ, is the amount of shift caused by the electric field. If we assume, n, molecules in the volume V, then the total dipole momentum is

$$\overline{p} = \sum_{i=1}^{n} q\overline{r_i^+} - \sum_{i=1}^{n} q\overline{r_i^-} = q\sum_{i=1}^{n}\left(\overline{r_i^+} - \overline{r_i^-}\right) \qquad (3.55)$$

We know from equation 3.54 that $\overline{r_i^+} - \overline{r_i^-} = \delta\hat{z}$. Hence,

$$\overline{p} = q\sum_{i=1}^{n} \delta\hat{z} = nq\delta\hat{z} \qquad (3.56)$$

When dealing with a large number of molecules or charges, we typically use density. For example, if the medium has N molecules per unit volume $N = n/V$, then the charge density in this case $\rho = Nq$ and the dipole momentum per unit volume is

$$\overline{P} = Nq\delta\hat{z} \qquad (3.57)$$

This vector is commonly referred to as **the polarization vector** and it has units of C/m^2. The polarization vector indicates how the medium responds to the applied electric field. This response is in the terms of the amount of shift induced by the field. As one can expect, the shift would depend strongly on what atoms form the medium and their arrangement as well as the strength of the applied electric field.

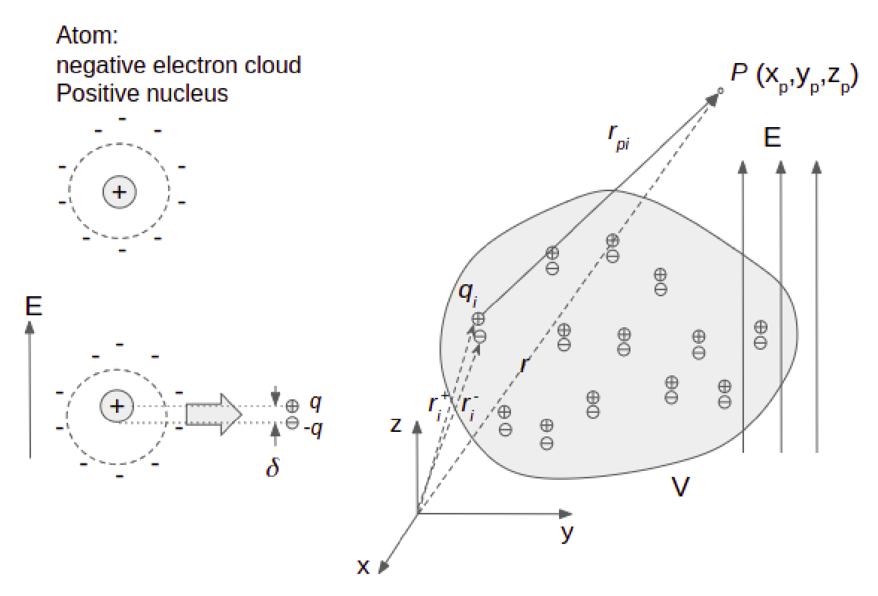


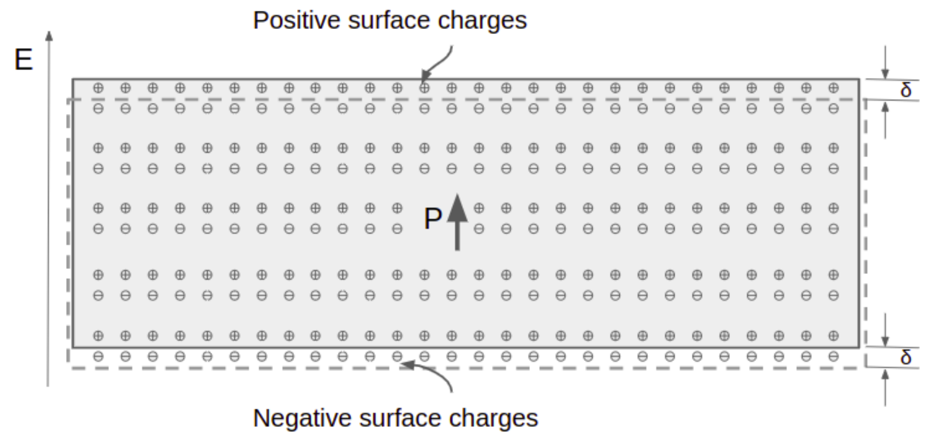
Figure 3.7. To the left: Dielectric medium atom shift of charges centers under an applied uniform electric field. To the right: A volume of dielectric medium under the application of a uniform electric field represented by a distribution of electric dipoles.

With that in mind, we can always argue that the polarization vector should be proportional to the applied electric field. This is because when there is no field applied the total polarization is zero ($\delta = 0$). Increasing the amplitude of the electrical field increases the shift in the center of charges and hence the polarization vector. In other words we can say that

$$\overline{P} = \alpha\overline{E} \qquad (3.58)$$

Where α is the proportionality factor and it is commonly known as the **electric polarizability**. This factor only depends on the material as it is separated from the electric field. Let us now consider a slab of a dielectric material of a finite thickness d and large extension in x and y directions. When an electric field along the z axis is applied on the mediu, the polarization vector is produced in the medium due to the displacement of the charge centers in the molecules as visualized in figure 3.8.

Figure 3.8. Charges displacement in a dielectric slab as a response to an applied uniform electric field along the z axis. That generates surface charges at the top and bottom layers.



From the visualization in figure 3.8 one can deduce that the total charges inside the dielectric slab add to zero except for two layers at the top and the bottom. Each layer has a thickness of the charges displacement distance δ. As this thickness is very small, then we could consider surface charge densities at these two layers that equals the volume charge density, Nq, multiplied by the layer thickness, or $\sigma_P = N\delta q$. Compared to equation 3.57, the surface charge density due to the separation of charges equals the amplitude of the polarization vector.

$$\sigma_P = |\overline{P}| \qquad (3.58)$$

To be more accurate, the equation above should be $\sigma_P = \overline{P} \cdot \widehat{n}$ where $\widehat{n}$ is a unit vector normal to the surface. In this case both $\widehat{n}$ and $\overline{P}$ are in the same direction and hence the dot product equals the amplitude of the polarization vector. So, when applying an electric field, the total charge inside the volume of the dielectric medium is zero and charges only exist on the surface in a fashion similar to charged metallic medium. This is however for a different reason here. In the metallic medium free charges are moved to the surface. In the dielectric medium however, bounded charges are causing the presence of the surface charges.

Capacitor with dielectric filling

Let us now place this dielectric medium inside a parallel plate capacitor which has a thickness L that is slightly larger than the slab as shown in figure 3.9. We know from before that the amplitude of the electric field produced by a charged metallic plate is $E = \sigma/\varepsilon_o$ when no medium is present. Here, σ is the surface charge density on the conductor surface due to free charges in metal. When the dielectric medium is placed, the electric field produced by the plates generates a polarization vector $\overline{P}$ inside the medium. This results in surface charge density σ_P, due to the bounded charges, with an opposite sign of σ. This is due to the fact that the electric field lines start from positive and end at negative charges.

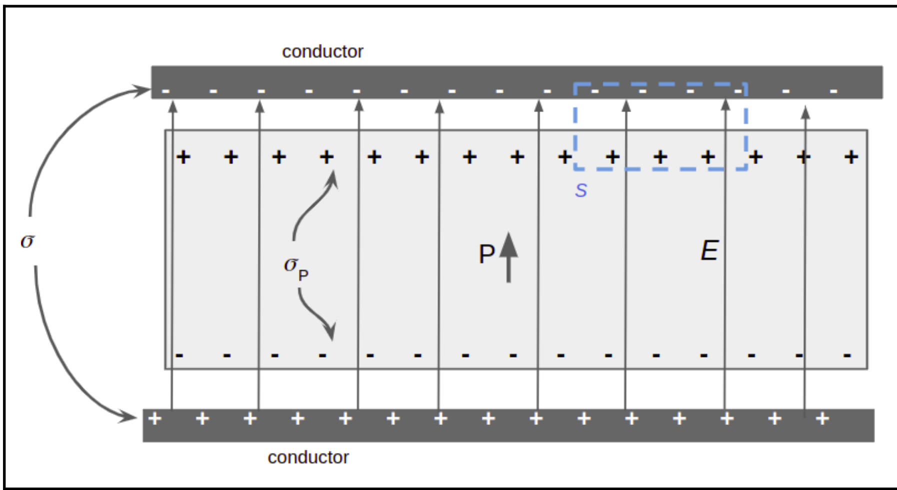


Figure 3.9. A dielectric slab between two charged parallel plates (capacitor). The electric field produces a polarization vector inside the medium. That causes a reduction of E compared to no dielectric medium.

In order to find the electric field we use our knowledge of the flux through a closed surface S (shown by the dashed lines in the figure.) We know that the flux through the surface S is $\phi_E = Q/\varepsilon_o$ where Q is the total charge inside the surface. If the top and bottom slides of S has an are A, then the total charge $Q = \left(\sigma - \sigma_P\right).A$. We also know that the flux equals the integration of the electric field over the surface area. Here, the electric field is normal to two surfaces, top and bottom. Then

$$\phi_E = \left(\sigma - \sigma_P\right).A/\varepsilon_o = E \cdot A \rightarrow E = \left(\sigma - \sigma_P\right)/\varepsilon_o \qquad (3.59)$$

We notice form equation 3.59 that the electric field is reduced due to the presence of the dielectric medium in comparison to the case with no medium where $E = \sigma/\varepsilon_o$.

We know that the bounded charge density equals the amplitude of the polarization vector, hence we can write equation 3.59 as

$$E = (\sigma - P)/\varepsilon_o = (\sigma - \alpha E)/\varepsilon_o \qquad (3.60)$$

Hence

$$E \cdot \left(1 + \alpha/\varepsilon_o\right) = \sigma/\varepsilon_o \qquad (3.61)$$

The term α/ε_o is defined as the **dielectric susceptibility** of the medium, $\chi = \alpha/\varepsilon_o$. The susceptibility is a unitless factor. The electric field is then

$$E = \frac{\sigma}{(1+\chi)\varepsilon_o} \qquad (3.62)$$

The term $1 + \chi$ is commonly known as the **relative permittivity** of the medium or $\varepsilon_r = 1 + \chi$. Then the electric field expression can be reduced to

$$E = \frac{\sigma}{\varepsilon_r \varepsilon_o} \qquad (3.63)$$

The product of the relative permittivity by the vacuum permittivity is known as the medium permittivity or $\varepsilon = \varepsilon_r \varepsilon_o$ and the electric field now is

$$E = \sigma/\varepsilon \quad (3.64)$$

In the case of vacuum, the susceptibility equals zero due to the absence of any medium inside the vacuum, and hence, the relative permittivity of vacuum is one. The presence of a medium results in a finite value of χ and hence the relative permittivity is typically greater than one, $\varepsilon_r > 1$. Using this finding, one can easily correct the capacitance value for parallel plates to

$$C = A\varepsilon/L \quad (3.64)$$

The capacitance hence increases when a dielectric medium is placed between the conductor plates. Hence, for a given stored charge of Q the electric potential is less when a dielectric medium is introduced, $C = Q/V_E$.

Dielectric medium and electric field divergence

In the beginning of this unit we carried a derivation of the electric field divergence through integration over an infinitesimal closed surface with uniform charge density ρ. The electric flux was

$$\psi_E = \int_S \overline{E} \cdot \overline{ds} = \Delta x \Delta y \Delta z (\overline{\nabla}.\overline{E}) = Q/\varepsilon_o \quad (3.65)$$

This derivation was done assuming that the volume $V = \Delta x \Delta y \Delta z$ is very small such that the divergence of the electric field is constant inside it. This situation is not realistic and in general we should be carrying an integration of $\overline{\nabla}.\overline{E}$ over the volume *V*. Also, the infinitesimal surface vector $\overline{ds}$ is actually the infinitesimal surface area ds multiplied by the normal unit vector $\widehat{n}$, or $\overline{ds} = ds\,\widehat{n}$. The electric flux can be then written as.

$$\psi_E = \int_S \overline{E} \cdot \widehat{n}\, ds = \int_V \overline{\nabla}.\overline{E}\, dV = \int_V \rho/\varepsilon_o\, dV \quad (3.66)$$

In equation 3.66 we represented the total charge Q as an integration of the charge density over the volume, $Q = \int_V \rho/\varepsilon_o\, dV$. We can observe from the relation above that the closed surface integration of the dot product of a vector to the normal of the surface equals to an integration of the divergence of the vector over the volume the closed surface occupies. Mathematically we write that as $\int_S \overline{E} \cdot \widehat{n}\, ds = \int_V \overline{\nabla}.\overline{E}\, dV$. The other result form equation 3.66 is what we obtained earlier that $\overline{\nabla}.\overline{E} = \rho/\varepsilon_o$. However in that derivation we only considered free charges. As we know by now that for a dielectric medium we need to consider two sets of charges: free and bounded charges (a result of the formation of the polarization vector). The electric field divergence in this case is

$$\overline{\nabla}.\overline{E} = \left(\rho + \rho_P\right)/\varepsilon_o \quad (3.67)$$

Here ρ is the volume charge density of the free charges and ρ_P is that for the bounded charges.

We stated earlier that the bounded surface charge density of the dielectric slab. Let's examine this statement when the polarization vector is not normal to the surface of the slab dielectric medium (not along the z axis.) In this case the charge displacement, $\overline{\delta}$, is along the direction of $\overline{P}$ as illustrated in figure 3.10. When estimating the surface charge density, σ_P, earlier we multiplied the charge volume density $\rho = Nq$ by the layer of thickness δ. That was fine when $\overline{P}$ was normal to the surface, i.e. along $\widehat{n}$ direction. In our situation here, the layer thickness d is smaller as it is the projection of $\overline{\delta}$ over the normal vector $\widehat{n}$, or $d = \overline{\delta} \cdot \widehat{n}$. In this case the charge surface densit
y is

$$\sigma_p = Nq\overline{\delta} \cdot \widehat{n} = \overline{P} \cdot \widehat{n} \tag{3.67}$$

As illustrated in figure 3.10, for every charge built on the surface an opposite charge is left inside the volume. That causes a finite charge density inside That has an opposite sign, $-\overline{P} \cdot \widehat{n}$. Even though the net charge inside the volume is zero but we can always argue that this displacement of charges across the surface causes a finite volume charge density inside. We can write that as

$$\int_V \rho_P dV = -\int_S \sigma_P ds = -\int_S \overline{P} \cdot \overline{n}\, ds \tag{3.68}$$

Recall from equation 3.66 that the integration of the dot product of a vector with the normal equals volume integration of the vector divergence.

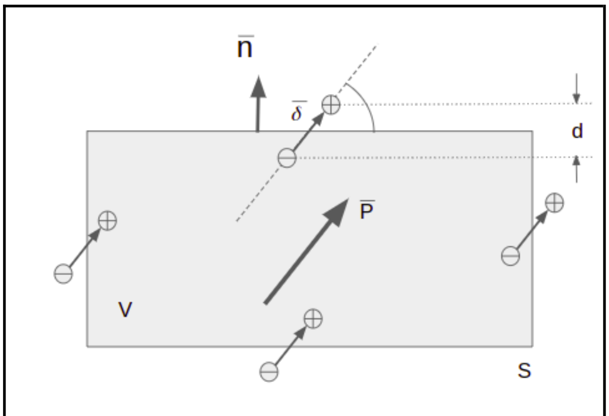


Figure 3.10. The case when the Polarization vector is not normal to the surface of the slab dielectric.

$$\int_V \rho_P dV = -\int_V \overline{\nabla} \cdot \overline{P}\, dV \tag{3.70}$$

Equating the terms inside the volume integration, we obtain a relation for the bounded charge volume density

$$\rho_P = -\overline{\nabla} \cdot \overline{P} \tag{3.71}$$

The electric field divergence in equation 3.67 becomes

$$\overline{\nabla}.\overline{E} = (\rho - \overline{\nabla}.\overline{P})/\varepsilon_o \tag{3.72}$$

We as well stated earlier that the polarization vector is proportional to the electric field through the polarizability factor, $\alpha = \varepsilon_o \chi$.

$$\overline{\nabla}.[(1 + \chi)\overline{E}] = \rho/\varepsilon_o \rightarrow \overline{\nabla}.\left[\varepsilon_r \overline{E}\right] = \rho/\varepsilon_o \tag{3.73}$$

If we assume that the relative permittivity is homogeneous, e.g. not varying in space, then

$$\overline{\nabla}.\overline{E} = \rho/\varepsilon_r\varepsilon_o = \rho/\varepsilon \tag{3.74}$$

As we might have expected, the divergence of the electric field in the case of homogeneous dielectric medium equals the free charges volume density divided by the medium permittivity, $\varepsilon = \varepsilon_r\varepsilon_o$.

Finally, let us revisit the electric displacement vector we introduced in unit two, equation 2.10. In the absence of a dielectric medium the displacement vector is $\overline{D} = \varepsilon_o \overline{E}$. By now, we can right away state that in the presence of a dielectric medium, this vector has to be modified as

$$\overline{D} = \varepsilon_r \varepsilon_o \overline{E} = \varepsilon \overline{E} \quad (3.75)$$

In this way we could rearrange equation 3.73 as

$$\overline{\nabla}.\left[\varepsilon_r \varepsilon_o \overline{E}\right] = \rho \rightarrow \overline{\nabla}.\overline{D} = \rho \quad (3.76)$$

Summary of electrostatics equations

description	Without dielectric	With dielectric
Potential	$\overline{E} = \overline{\nabla}\phi$	$\overline{E} = \overline{\nabla}\phi$
Displacement vector	$\overline{D} = \varepsilon_o \overline{E}$	$\overline{D} = \varepsilon \overline{E}$
Divergence	$\overline{\nabla}.\overline{E} = \rho/\varepsilon_o$ $\overline{\nabla}.\overline{D} = \rho$	$\overline{\nabla}.\overline{E} = \rho/\varepsilon$ $\overline{\nabla}.\overline{D} = \rho$
Curl	$\overline{\nabla} \times \overline{E} = 0$	$\overline{\nabla} \times \overline{E} = 0$
Parallel plate capacitance	$C = A\varepsilon_o/L$	$C = A\varepsilon/L$

3d. Electrostatic energy

Force, work and potential

In the first unit we introduced Coulomb's law that describes the force between two charges q_1 and q_2 that are separated by a distance r_{12}, $\overline{F} = \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}^2}\hat{r}_{12}$. Here $\hat{r}_{12}$ is a unit vector pointing from p_1 to p_2. If we set the center of the coordinates to point p_1 and consider the field generated by q_1 then in this system a work is required to bring the charge q_2 to a distance r_{12} against the force it experiences.

$$W = -\int_{\infty}^{p_2} \overline{F}.\overline{dr} = \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}} = q_2\left(\frac{q_1}{4\pi\varepsilon_o r_{12}}\right) \quad (3.77)$$

In other words, in a system of two charges there is an energy, $U = W$, needed to bring these two charges together. The term in the parentheses in equation 3.77 represents the electrical potential caused by charge q_1, which is the work needed to bring a unit charge from infinity to a distance r_{12}. The energy in a system of the two charges is $U_{12} = q_2 \cdot \phi_{12}$. We could arrange equation 3.77 differently so that $q_1$is outside the parentheses and in this case the energy is $U_{21} = q_1 \cdot \phi_{21}$. Here, the notations are used such that

$$\phi_{ij} = \frac{q_i}{4\pi\varepsilon_o r_{ij}} \quad (3.78a)$$

$$U_{ij} = q_j \cdot \phi_{ij} = q_i \cdot \phi_{ji} = U_{ji} \quad (3.78b)$$

System of multiple charges

In the previous section, we showed that a system of two charges has an electrostatic energy that equals one charge multiplied by the potential caused by the other. If the system contains more than just two charges, then the total energy is the sum of all energies between each pair of charges that could be possibly formed.

$$U_{total} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} U_{ij} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{q_i q_j}{4\pi\varepsilon_o r_{ij}} \quad (3.79)$$

As you can notice in the summation, the index *i* varies from 1 to N-1 where N is the total number of charges. The index *j* however varies from *i*+1 to N. This gives us all the possible combinations without counting the energy of a pair twice as we already know that $U_{ij} = U_{ji}$. When the number of charges becomes very large, one would need to perform integration instead of the summation. Obtaining a closed form solution is difficult for arbitrary distribution of charges. However, let us benefit from special cases of distribution and revisit the sphere of charges example in unit two (figure 2.5). The total sphere radius is r_s and it has a uniform charge density ρ C/m^3. We know from equation 2.20 that the electric field inside the volume due a smaller sphere of radius r is

$$\overline{E} = \frac{Q}{4\pi\varepsilon_o r^2}\hat{r} \quad (3.80)$$

The electric potential to bring a unit charge from infinity to the radius r is then $\phi = \frac{Q}{4\pi\varepsilon_o r}$. The electrostatic energy between the charge and the small sphere is $q\phi$. To consider all the charges within the thin layer, the charge q is then replaced by $4\pi r^2 dr\, \rho$. The energy in the system to bring the spherical layer and the small sphere is then

$$dU = 4\pi r^2 dr\, \rho \frac{Q}{4\pi\varepsilon_o r} \quad (3.81)$$

Where $Q = 4\pi r^3 \rho/3$. Then

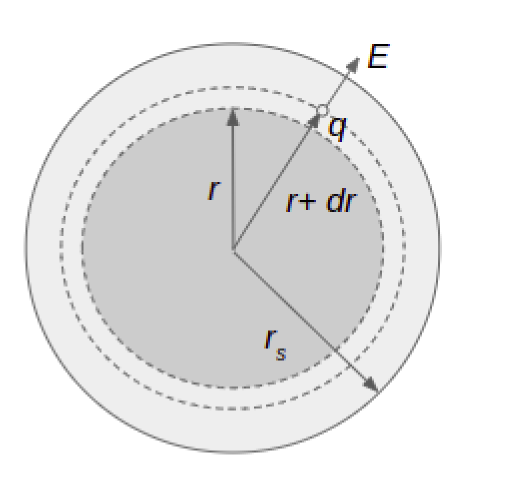


Figure 3.11. Calculating the energy of a sphere of charges.

$$dU = \frac{4\pi}{3\varepsilon_o}\rho^2 r^4 dr \quad (3.82)$$

The total energy needed to bring all the charges in the sphere is calculated by integrating equation 3.82 from *r* = 0 to the radius of the sphere, r_s.

$$U = \int_{r=0}^{r_s} \frac{4\pi}{3\varepsilon_o}\rho^2 r^4 dr = \frac{4\pi}{15\varepsilon_o}\rho^2 r_s^5 \quad (3.82)$$

In terms of the total charge of the sphere, $Q_s = \frac{4\pi}{3} r_s^3 \rho$, we can rewrite equation 3.82 as

$$U = \frac{3}{5}\frac{Q_s^2}{4\pi\varepsilon_o r_s} \quad (3.83)$$

Energy of charged conductor

In the case of conductors, charges are present at the surface in form of surface charge density σ where the produced electric field from any point at the surface is $\overline{E} = \frac{\sigma}{\varepsilon_o}\hat{n}$. In the case of a conducting sphere, the unit vector $\hat{n}$ is along the radial direction, $\overline{E} = \frac{\sigma}{\varepsilon_o}\hat{r}$. In terms of the total charge, $\overline{E} = \frac{Q}{4\pi r_x^2 \varepsilon_o}\hat{r}$ and the electric potential is $\phi = V_E = \frac{Q}{4\pi r_s \varepsilon_o}$. The energy needed to bring an infinitesimal charge dQ to the surface is

$$dU = dQ\phi = \frac{Q}{4\pi r_s \varepsilon_o} dQ \quad (384)$$

The total energy required to bring all the charges on the sphere is then

$$U = \int \frac{Q}{4\pi r_s \varepsilon_o} dQ = \frac{Q^2}{8\pi r_s \varepsilon_o} \quad (385)$$

Recall that the self capacitance of a charged sphere is $C = 4\pi\varepsilon_o r_s$. We can then write the energy of the conductive sphere in terms of its self capacitance as

$$U = \frac{Q^2}{2C} \quad (3.86)$$

In terms of the potential, we know that $Q = \phi/C$ then $U = \frac{1}{2}C\phi^2$. Let us recall here that it is common to refer to the electric potential difference as V. The electrostatic energy of the capacitor system as

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C} \quad (3.87)$$

The relation in equation 3.87 is actually a general form and it is not restricted to the charged sphere. If we recall that the energy needed to bring the infinitesimal charge dQ is $dU = \phi dQ$ and we use the definition of the capacitance, $C = Q/\phi$, then we can write

$$dU = QdQ/C \quad (3.88)$$

Performing the integration, one obtains the expression in equation 3.86. Table 3.1 shows the electrostatic energy of several know capacitors

Geometry	**Capacitance**	**Energy**
sphere	$4\pi r_s \varepsilon_o$	$\frac{Q^2}{8\pi r_s \varepsilon_o} = 2\pi r_s \varepsilon_o V^2$
Two parallel sheets	$A\varepsilon_o/L$	$\frac{L\,Q^2}{A\varepsilon_o} = \frac{1}{2}A\varepsilon_o V^2/L$
Concentric cylinders	$2\pi\, h\, \varepsilon_o/ln\left(\frac{r_2}{r_1}\right)$	$\frac{Q^2 ln\left(\frac{r_2}{r_1}\right)}{4\pi\, h\, \varepsilon_o} = \pi\, h\, \varepsilon_o V^2/ln\left(\frac{r_2}{r_1}\right)$
Concentric spheres	$4\pi\varepsilon_o/\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$	$\frac{Q^2}{8\pi\varepsilon_o}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 2\pi\varepsilon_o V^2/\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

T**able 3.1.** Summary of energy in several capacitors