Unit Five:

More on magnetostatics

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5a. Current flowing through a volume

In the previous analysis we considered a thin conductive wire where the moving charges were assumed to have a constant linear charge density. Now, let us consider the volume effect as in figure 4.10. The electrical current *I* inside the volume consists of a collection of charges, with charge density *N* charges per unit volume, that move with the same velocity $\overline{v}$ along the wire that has a cross sectional area ΔS. Hence the volume of the segment of length *dl* in figure 4.10 is $\Delta V = \Delta S\, dl$. The force exerted on each charge inside ΔV is $\overline{F}_j = q_j \overline{v}_j \times \overline{B}$. The total force on the segment is summation of all the forces inside the segment

$$d\overline{F} = \sum_{j=1}^{n} \overline{F}_j \qquad (5.1)$$

Where *n* is the total number of charges inside ΔV, which is $n = N\Delta V$. If all the charges have the same amplitude and velocity then

$$d\overline{F} = \sum_{j=1}^{N\Delta V} q\ \overline{v}\ \times \overline{B} \qquad (5.2a)$$

$$= (N\Delta V)(q\overline{v} \times \overline{B}) \qquad (5.2b)$$

We can arrange equation 5.2b as

$$d\overline{F} = (Nq\overline{v}) \times \overline{B}\Delta V \qquad (5.3)$$

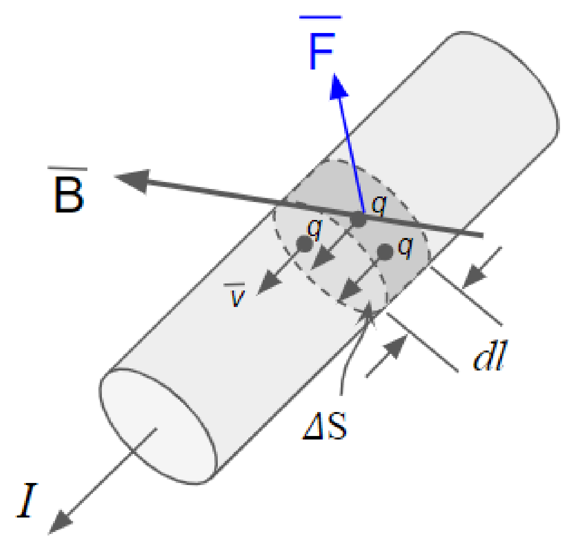


Figure 5.1

So what is that term $(Nq\overline{v})$ in equation 5.3?

Current density

Let us zoom on the wire in figure 5.1 and examine the flow of charges in time. If we still assume that all charges are moving at same velocity along the wire axis, then we could state that within a time duration *dt*, all charges in a volume $\Delta V = \Delta S\, v\, dt$ will cross the surface S_1 as shown in figure 5.2. The total charge that flows through the surface within time *t* is then

$$dQ = N\,\Delta V q = N\,(\Delta S\, v\, dt)\, q \qquad (5.4)$$

The current that follows can be approximated as

$$I = \frac{dQ}{dt} = N\,\Delta S\, v\ q \qquad (5.5)$$

The amount of current that flows through a unit area is known as the **current density** and it is defined as

$$j = \frac{I}{\Delta S} = N\, v\ q \qquad (5.6)$$

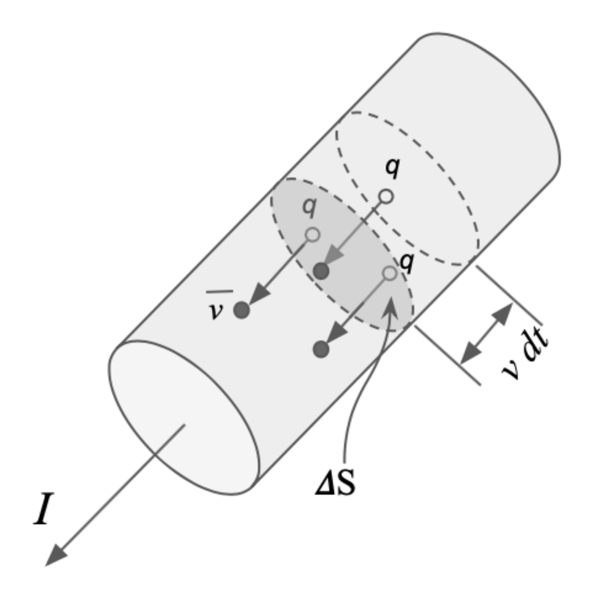


Figure 5.2. Charges flowing through a surface ΔS from a volume $\Delta V = vt\,\Delta S$ during a time *t*.

and that is exactly the term in parentheses in equation 5.3. The current density as a vector has the same direction as the current and hence equation 5.6 can be written in a vector form as

$$\bar{j} = N\bar{v}\,q \tag{5.7}$$

And the magnetic force on the charges within the volume ΔV is

$$d\bar{F} = \bar{j} \times \bar{B}\Delta V \tag{5.8}$$

We know that $\Delta V = \Delta S\,dl$, then

$$d\bar{F} = \Delta S\,\bar{j} \times \bar{B}dl = \bar{I} \times \bar{B}dl \tag{5.9}$$

The force per unit length is then

$$\frac{d\bar{F}}{dl} = \bar{I} \times \bar{B} \tag{5.10}$$

Current flow through arbitrary surface

For the case of a current flowing through a cylindrical shape wire with a cross sectional area ΔS, the relation between the current and the current density is simply $\bar{I} = \bar{j}\Delta S$. That was due to the fact that the current density is uniform and that it flows normal to the cross section. Let us consider now a case where the wire is cut at angle θ as in figure 5.3.

For the scenario in figure 5.3, the current that flows through the tilted edge is exactly the same current that flows through a normal surface with a reduced area as shown by the dashed lines in the figure. In this case, we can write the total current as

$$\bar{I} = \bar{j}\Delta S\,cos\theta \tag{5.11}$$

The above equation can be written in terms of vector operation as

$$\bar{I} = \bar{j}.\hat{n}\,\Delta S\,\hat{e}_j \tag{5.12}$$

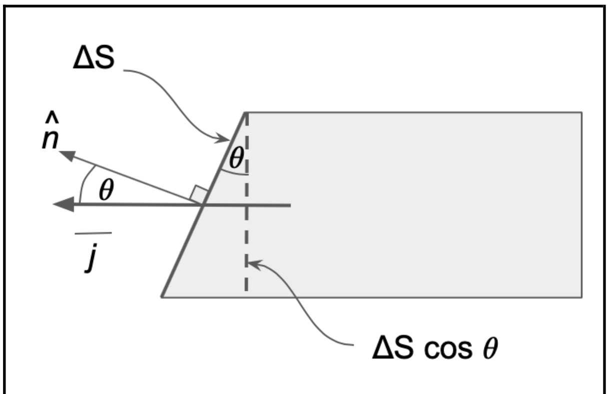


Figure 5.3. Side view of a current flowing in wire with an edge that is cut at an angle.

In equation 5.12, $\hat{n}$ is a unit vector normal to the surface ΔS and $\hat{e}_j$ is a unit vector along the direction of flow of the current density. The dot product $\bar{j}.\hat{n} = |\bar{j}|\,cos\theta$ produces the cosine term. Hence $\bar{j}.\hat{n}\,\Delta S\,\hat{e}_j = |\bar{j}|\,cos\theta\Delta S\,\hat{e}_j = \bar{j}\Delta S\,cos\theta$, which is the same expression in equation 5.11. Using this approach, an arbitrary surface S can be divided into small surface elements dS, as shown in figure 5.4, through each an elementary current *dI* flows.

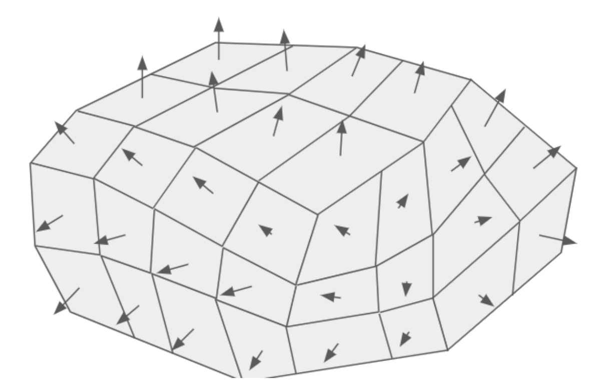


Figure 5.4. Breaking an arbitrary closed surface S into elementary surfaces each with a different normal unit vector.

The amplitude of the elementary current is

$$dI = \bar{j}.\hat{n}\, ds \qquad (5.13)$$

where $\hat{n}$ is a unit vector normal to the elementary surface area *ds*. The total current is then calculated by integrating dI over the surface S

$$I = \int_S \bar{j}.\hat{n}\, ds \qquad (5.14)$$

Conservation of charges

We know from equation 5.5 that the current is the rate of change of charges per unit time $I =- \frac{dQ}{dt}$. Notice that a negative sign is introduced here. That is due to the fact that when the current is flowing out the closed surface, the total charge is expected to reduce and hene the rate of change of Q with time is negative. Hence, a negative sign is introduced in order to produce a positive current. The total charge inside the closed surface S can be expressed by a volume integration over the charge density ρ.

$$I =- \frac{dQ}{dt} =- \frac{d}{dt}\int_V \rho dV =- \int_V \frac{d\rho}{dt} dV \qquad (5.15)$$

Replacing the current with the expression 5.14.

$$\int_S \bar{j}.\hat{n}\, ds =- \int_V \frac{d\rho}{dt} dV \qquad (5.16)$$

This is known as the **conservation of charges**. Its states that *the total amount of electric charge in a system does not change with time.* So, the charges change when there is a current flowing in or outside the system which is the closed surface in our case. Let us now follow a similar approach in unit three to find a differential expression for equation 5.16. Consider an infinitesimal volume $dV = dxdydz$ as shown in figure 5.4. The volume has six faces and hence the total surface area is composed of six surfaces with six normal unit vectors as listed in table 5.1. The dot product of the current density and the normal vector at each surface is listed in the table. Notice that the current density in general varies in space and

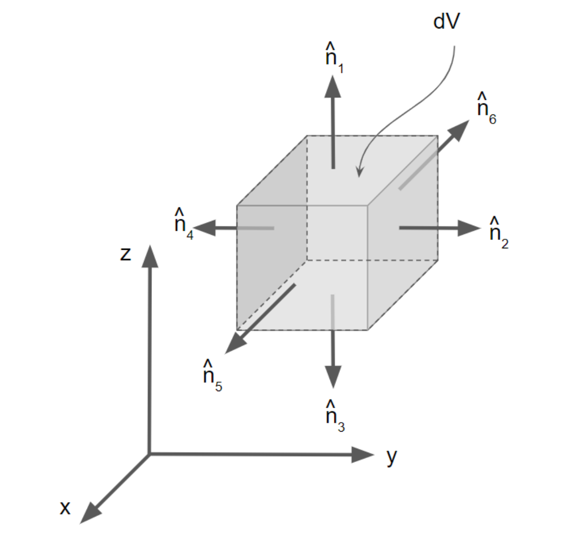


Figure 5.5. Infinitesimal volume, dV inside a closed surface ds with six surfaces.

hence the position dependency is added for each dot product to indicate the location of the surface plane. For instance, if the current density at a point at surface 1 is $j(x, y, z)$, then the density at an equivalent point at surface 3 is $j(x, y, z - dz)$ as surface 3 is at a plane that is separated from surface 1 by *dz*.

Table 5.1. Surfaces of dV

Surface	$\hat{n}$	ds	$\bar{j} \cdot \hat{n}\, ds$
1	$\hat{z}$	$dxdy$	$j_z(x, y, z)dxdy$
2	$\hat{y}$	$dxdz$	$j_y(x, y, z)dxdz$
3	$-\hat{z}$	$dxdy$	$-j_z(x, y, z - dz)dxdy$
4	$-\hat{y}$	$dxdz$	$-j_y(x, y - dy, z)dxdz$
5	$\hat{x}$	$dydz$	$j_x(x, y, z)dydz$
6	$-\hat{x}$	$dydz$	$-j_x(x - dx, y, z)dxdy$

The total dot product is then

$$\bar{j}.\hat{n}\, ds = j_x(x, y, z)dydz - j_x(x - dx, y, z)dydz + j_y(x, y, z)dxdz - j_y(x, y - dy, z)dxdz + j_z(x, y, z)dxdy - j_z(x, y, z - dz)dxdy \quad (5.17)$$

The expression in 5.17 can be simplified as

$$\bar{j}.\hat{n}\, ds = \frac{j_x(x,y,z)-j_x(x-dx,y,z)}{dx}dxdydz + \frac{j_y(x,y,z)-j_y(x,y-dy,z)}{dy}dxdydz + \frac{j_z(x,y,z)-j_z(x,y,z-dz)}{dz}dxdydz \quad (5.18)$$

Notice that the term $\frac{j_x(x,y,z)-j_x(x-dx,y,z)}{dx}$ is a valid approximation of the derivative of the current density with respect to x when dx is very small, or $\frac{dj_x}{dx} \approx \frac{j_x(x,y,z)-j_x(x-dx,y,z)}{dx}$. Hence, the expression in 5.18 becomes

$$\bar{j}.\hat{n}\, ds = \left(\frac{dj_x}{dx} + \frac{dj_y}{dy} + \frac{dj_z}{dz}\right)dxdydz \quad (5.19)$$

We know from before that $\bar{\nabla} \cdot \bar{j} = \frac{dj_x}{dx} + \frac{dj_y}{dy} + \frac{dj_z}{dz}$ and $dV = dxdydz$, then

$$\bar{j}.\hat{n}\, ds = \bar{\nabla} \cdot \bar{j}\, dV \quad (5.20)$$

The total current can now be obtained by integrative over the volume

$$\int_S \bar{j}.\hat{n}\, ds = \int_V \bar{\nabla} \cdot \bar{j}\, dV = -\int_V \frac{d\rho}{dt} dV \qquad (5.21)$$

The surface integral over S is now transferred to volume integration. We can then equate the terms inside the integration as follows

$$\bar{\nabla} \cdot \bar{j} = -\frac{d\rho}{dt} \qquad (5.22)$$

This is another way to write the conservation of charges where the change of the charge density is due to increase or decrease in the current flow through the system.

5b. Steady current

When the current is not varying with time, we commonly refer to it as steady current or DC current. This is the case of magnetostatics limit.

Gauss's law

A steady current that flows through a wire generates a magnetic field as in figure 5.6. Let us consider a closed cylindrical surface as shown by the dashed lines in the figure. In this configuration, the magnetic field flows inward normal to S_1 and outwards normal to S_2. The field lines are parallel to S_3. Hence, we can write the following relation

$$\int\int_S \bar{B} \cdot \hat{n}\, ds = BS_1 - BS_2 + 0 \qquad (5.23)$$

For the cylindrical surface, both S_1 and S_2 are equal and hence

$$\int\int_S \bar{B} \cdot \hat{n}\, ds = 0 \qquad (5.24)$$

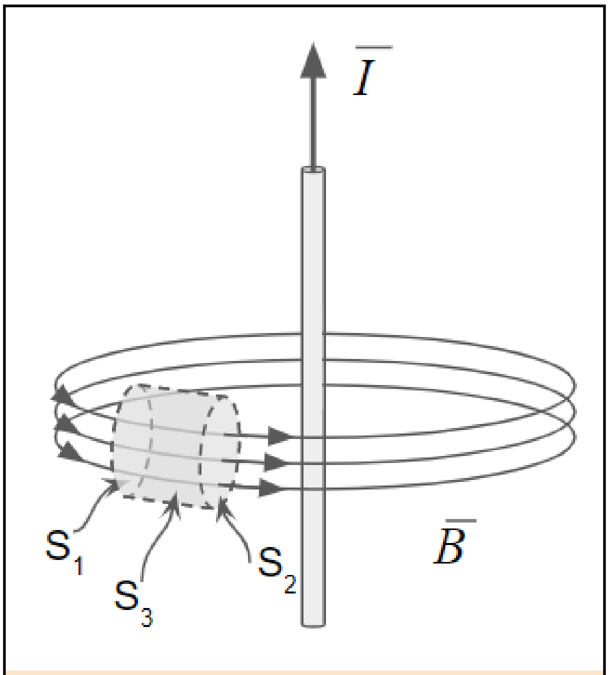


Figure 5.6. Gauss law for magnetism

The expression in 5.24 is known as **Gauss's law for magnetism**. We can write equation 5.24 in the differential form as we did in equation 5.22 as

$$\bar{\nabla} \cdot \bar{B} = 0 \qquad (5.25)$$

Oersted-Ampere's law

The magnetic field produced by a steady current I is estimated from Biot-Savart law as in equation 4.22

$$\overline{B} = \frac{\mu_o I}{2\pi R}\,\hat{\theta} \tag{5.26}$$

Performing an integration along a loop of radius R we obtain

$$\oint_l \overline{B}.\overline{dl} = \int_{\theta=0}^{2\pi} \frac{\mu_o I}{2\pi R}\,Rd\theta \tag{5.27}$$

For a steady or DC current, the integration above becomes

$$\oint_l \overline{B}.\overline{dl} = \mu_o I \tag{5.28}$$

This is known as **Oersted's circuital law** or **Ampere's circuital law for steady current**. The law states the following rules regarding magnetic field and current in straight wires:

- The magnetic field lines encircle the current-carrying wire *(along $\hat{\theta}$ direction)*.
- The magnetic field lines lie in a plane perpendicular to the wire *(The current is along $\hat{z}$ and magnetic field in the x-y plane)*.
- If the direction of the current is reversed, the direction of the magnetic field reverses.
- The strength of the field is directly proportional to the magnitude of the current.The strength of the field at any point is inversely proportional to the distance of the point from the wire.

These rules clearly describe the expression in equation 5.26 that is driven for a magnetic field produced by a steady current flowing in the positive z-direction.

If we express the current in terms of current density as in equation 5.14, equation 5.28 is expanded to

$$\oint_l \overline{B}.\overline{dl} = \mu_o \int\int_S \overline{j}.\hat{n}\,ds \tag{5.29}$$

Using the relation in equation 3.31 we can write the line integral as

$$\oint_l \overline{B}.\overline{dl} = \int\int_S \overline{\nabla} \times \overline{B}.\hat{n}\,ds \tag{5.30}$$

Comparing equations 5.30 and 5.29, we can write

$$\overline{\nabla} \times \overline{B} = \mu_o \overline{j} \tag{5.31}$$

This is the differential form of Oersted-Ampere's law. In the following text we will be referring to this law as Ampere's law for short. It reads that the curls of the magnetic field equals the current density multiplied by the vacuum permeability. This is clearly presented where the magnetic field curls in a circular path around the current flowing in a straight wire.

Magnetic field in a coil

It is time now to benefit from the simplicity of Ampere's law to calculate the magnetic field produced by a current flowing in a more complicated geometry. Here, we consider a coil that is winded in a solonidal form as shown in figure 5.7.

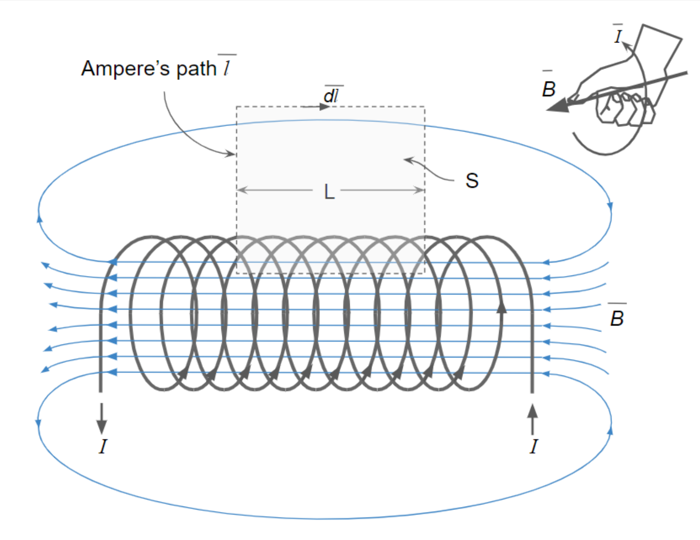


Figure 5.7. Magnetic field generated by a steady current flowing in a coil that forms a solenoid path.

Notice that in the figure, the direction of the magnetic field follows the right hand rule where the current circulates with the four fingers and the field is along the direction of the thumb.

Let us now select a rectangular path $\bar{l}$ that has a width L as shown by the dashed lines in the figure. From Ampere's law in equation 5.28 we know that the integration of the dot product of the magnetic field and the path $\bar{l}$ equals the total current that flows the area S. So, what is the total current that flows through S?

We know that the current is flowing in multiple loops formed by the coil. Hence, everytime the coil intersects the surface S, there is a current of amplitude I is flowing through. Hence, if the coil crosses the surface N times, the total current that flows through the surface is then $I_{total} = NI$. we can now write Ampere's equation as

$$\oint_l \bar{B}.\overline{dl} = \mu_o I_{total} = \mu_o NI \quad (5.32)$$

For the left side of the equation above, we can observe from the figure that the magnetic field is almost perpendicular to the two vertical sides of the path. Hence, the dot product between the field and $d\bar{l}$ along these two sides is negligible. The field is however almost parallel to the two horizontal sides of the path and hence

$$\oint_l \bar{B}.\overline{dl} = BL + B_{far}L \quad (5.33)$$

Where B_{far} is the magnetic field at the far side of the path. We know from Biot-Savart law that the field is inverse proportional to the distance from the loop. So, if we select the length of the path to be large enough then the magnitude of B_{far} becomes negligible and we can state that

$$BL = \mu_o NI \rightarrow B = \mu_o nI \quad (5.34)$$

where $n = N/L$ is the number of loops per unit length.

Summary of magnetostatic limit

The following table summarizes the main equations that govern the behavior of moving charges in magnetostatics limit

Table 5.2. Summary of the main laws in magnetostatics limit

Law	**equation**	**Other possible form**
Lorentz force/ electromagnetic force	$\overline{F} = q\left(\overline{E} + \overline{v} \times \overline{B}\right)$	
Biot-Savart law	$\overline{B} = \oint_l \frac{\mu_o I}{4\pi r^2}\left(\hat{l} \times \hat{r}\right)dl$	
Ampere's force law	$\frac{dF_B}{dl} = \mu_o I_1 I_2 / 2\pi R$	
Conservation of charges	$\overline{\nabla} \cdot \overline{j} = -\frac{d\rho}{dt}$	$\int_S \overline{j}.\hat{n}\,ds = -\int_V \frac{d\rho}{dt}dV$
Gauss's law of magnetism	$\overline{\nabla} \cdot \overline{B} = 0$	$\int\int_S \overline{B} \cdot \hat{n}\,ds = 0$
Oersted-Ampere's law	$\overline{\nabla} \times \overline{B} = \mu_o \overline{j}$	$\oint_l \overline{B}.\overline{dl} = \mu_o I$ $\oint_l \overline{B}.\overline{dl} = \mu_o \int\int_S \overline{j}.\hat{n}\,ds$

5.c. Magnetic dipole

In unit four, we discussed the magnetic field produced by a current loop using Biot-Savart law as presented in equations 4.30. Solving such integrals typically requires a numerical approach. However, one could obtain a simple solution for specific cases.

Magnetic moment

One case is to assume that the loop radius *R* is very small and the observation point is located far from the loop as in figure 5.8. In this case, $r_p \gg R$ and the expression for $1/r^3$ in equations 4.30 can be simplified using Taylor expansion as

$$1/r^3 = \frac{1}{\left(R^2 + r_p^2 - 2Rx_p\cos\theta - 2Ry_p\sin\theta\right)^{3/2}} \approx \frac{1}{r_p^3}\left(1 + 3R\left(x_p\cos\theta + y_p\sin\theta\right)\right) \qquad (5.35)$$

Using this expansion, the integrations in 4.30a for instance becomes

$$B_x \approx \frac{\mu_o I R z_p}{4\pi r_p^3}\int_{\theta=0}^{2\pi}\left(cos\theta + \frac{3R}{r_p^2}\left(x_p cos\theta^2 + y_p sin\theta cos\theta\right)\right)d\theta \quad (5.36)$$

Notice that in equation 5.36 that the only term that has a non-zero value in integrate is the term with $cos\theta^2$. Integration from 0 to 2π of $cos\theta^2$ is π and the magnetic field component in this case becomes

$$B_x \approx \frac{3\mu_o z_p x_p}{4\pi r_p^5}\left(I \cdot \pi R^2\right) \quad (5.36)$$

SImilarly, the y component of the magnetic field in equation 4.30b becomes

$$B_y \approx \frac{3\mu_o z_p y_p}{4\pi r_p^5}\left(I \cdot \pi R^2\right) \quad (5.37)$$

For the x component, the integration in equation 4.30c can be expanded as

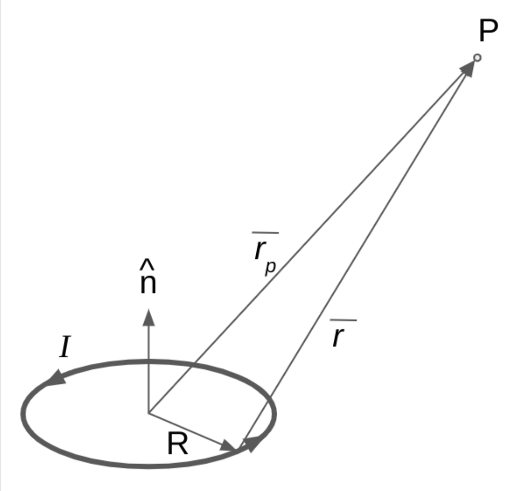


Figure 5.8. Magnetic field observed at a far distance is caused by a small current loop.

$$B_z \approx \frac{\mu_o I R}{4\pi r_p^3}\int_{\theta=0}^{2\pi}\left(1 + \frac{3R}{r_p^2}\left(x_p cos\theta + y_p sin\theta\right)\right)\left(R - x_p cos\theta - y_p sin\theta\right)d\theta \quad (5.38)$$

Expanding the multiplication term to

$$R + \frac{3R}{r_p^2}\left(x_p cos\theta + y_p sin\theta\right) - \left(x_p cos\theta + y_p sin\theta\right) - \frac{3R}{r_p^2}\left(x_p cos\theta + y_p sin\theta\right)^2 =$$

$$R + \frac{3R}{r_p^2}\left(x_p cos\theta + y_p sin\theta\right) - \left(x_p cos\theta + y_p sin\theta\right) - \frac{3R}{r_p^2}\left[\left(x_p cos\theta\right)^2 + 2x_p y_p cos\theta sin\theta + \left(y_p sin\theta\right)^2\right]$$

The expression above has one constant term, R, and two terms that have a squared cosine and sine. We know that integration from 0 to 2π of $cos\theta^2$ or $sin\theta^2$ gives us π. Hence, the integration in 5.38 gives us the following result

$$B_z \approx \frac{2\mu_o}{4\pi r_p^3}\left(I \cdot \pi R^2\right) - \frac{3\mu_o\left(x_p^2+y_p^2\right)}{4\pi r_p^5}\left(I \cdot \pi R^2\right) \quad (5.39)$$

One can directly observe that for the term in parentheses in equations 5.36, 5.37 and 5.39 the expression πR^2 is the area of the current loop and we can replace it with the symbol A. Also, we know that $r_p^2 = x_p^2 + y_p^2 + z_p^2$. So, we can complete the term in the parentheses in the second term in equation 5.39 as

$$B_z \approx IA \cdot \frac{2\mu_o}{4\pi r_p^3} - IA \cdot \frac{3\mu_o\left(x_p^2+y_p^2+z_p^2-z_p^2\right)}{4\pi r_p^5} = IA \cdot \frac{\mu_o}{4\pi r_p^3}\left(2 - 3 + \frac{z_p^2}{r_p^2}\right) = IA \cdot \frac{\mu_o}{4\pi r_p^3}\left(\frac{z_p^2}{r_p^2} - 1\right) \quad (5.40)$$

The magnetic field produced by the small loop is then

$$\overline{B} = IA \cdot \frac{\mu_o}{4\pi r_p^3} \cdot \left(\frac{3z_p x_p}{r_p^2}, \frac{3z_p y_p}{r_p^2}, \frac{3z_p z_p}{r_p^2} - 1 \right) \quad (5.41)$$

Equation 5.41 can be written as a subtraction of two vectors

$$\overline{B} = \frac{\mu_o}{4\pi r_p^3} \cdot \left[3IA\frac{z_p}{r_p} \cdot \left(\frac{x_p}{r_p}, \frac{y_p}{r_p}, \frac{z_p}{r_p} \right) - IA\hat{z} \right] \quad (5.42)$$

In equation 5.42, we took $\frac{z_p}{r_p}$ as a common factor from the first parentheses.The term remaining in the parentheses in 5.42 is the unit vector $\hat{r}_p$. For the loop defined in figure 5.8, the normal direction is along the z-axis or ,$\hat{n} = \hat{z}$. Hence, we can define a vector quantity

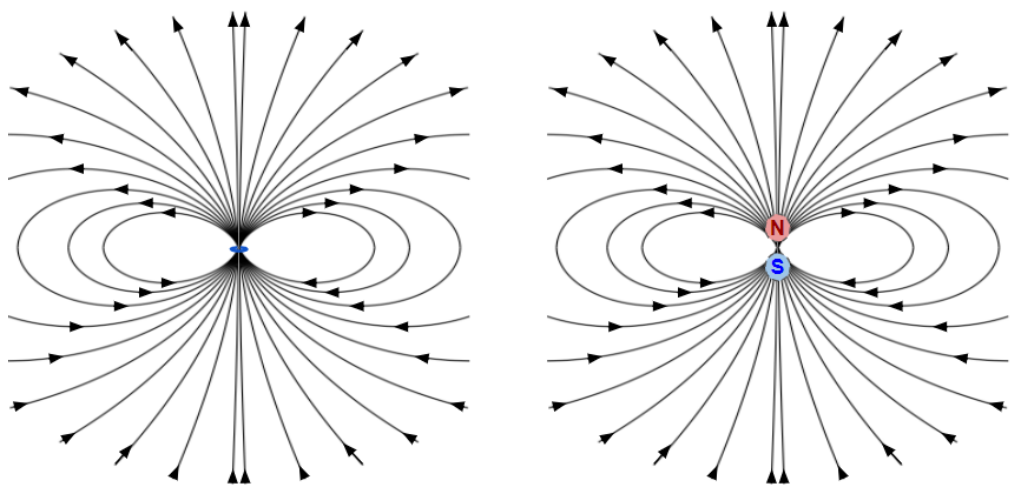
$$\overline{m} = IA\hat{n} \quad (5.43)$$

that is known as the **magnetic moment** and has units of A.m^2. Using this definition we can write the term $IA\frac{z_p}{r_p}$ as a dot product of the momentum and the displacement unit vector,

$$\overline{m} \cdot \hat{r}_p = IA\hat{z} \cdot \left(\frac{x_p}{r_p}, \frac{y_p}{r_p}, \frac{z_p}{r_p} \right) = IA\frac{z_p}{r_p}. \quad (5.44)$$

Hence we can now rewrite equation 5.42 as

$$\overline{B} = \frac{\mu_o}{4\pi r_p^3} \cdot \left[3(\overline{m} \cdot \hat{r}_p)\hat{r}_p - \overline{m} \right] \quad (5.45)$$



(a) (b)

Figure 5.9. (a) Magnetic field produced by a small current loop. (b) magnetic dipole representation of the current loop.

We can now draw the magnetic field lines due to a small current loop with current flowing anti-clockwise as shown in figure 5.9a. The magnetic field lines in the figure gives us a great remblence to that of the electric field produced by an electrical dipole (figure 3.5 in unit three) formed by a positive and a negative charge. Similarly we can represent the current loop by two poles. One pole is at the top from which the magnetic field lines are pointing outwards of the loop. We call that pole north and give it a short letter N. The second pole is at the bottom from which the lines are pointing inwards and we call it south or S for short. This representation is illustrated in figure 5.9b and it is commonly referred to as a **magnetic dipole** in analogy to the electrical dipole. In comparison, the electrical dipole moment is defined as $\overline{p} = q\overline{l}$ where $\overline{l}$ is the displacement vector pointing from the negative to the positive charge. However for the magnetic dipole, as we defined earlier, the magnetic moment is defined as the vector that has an amplitude equals the current in the loop multiplied by the loop area and a direction normal to the loop and following the right hand rule, $\overline{m} = IA\widehat{n}$.

Force on a current loop by an external magnetic field

Consider a rectangular current loop that is placed in the way of a constant magnetic field as shown in figure 5.10. For the configuration in the figure, the magnetic field is along the y direction. The magnetic force applied on a charge element q that is moving in the loop is $d\overline{F} = q\overline{v} \times \overline{B}$. We can write this in terms of current density as

$$d\overline{F} = \rho_l dl\, \overline{v} \times \overline{B} \qquad (5.46)$$

As defined earlier, the current in the loop is $\overline{I} = \rho_l \overline{v}$. Hence, we can rewrite equation 5.46 as

$$d\overline{F} = \overline{I} \times \overline{B} dl \qquad (5.47)$$

For the first vertical side, the current is moving along the positive z-direction, hence

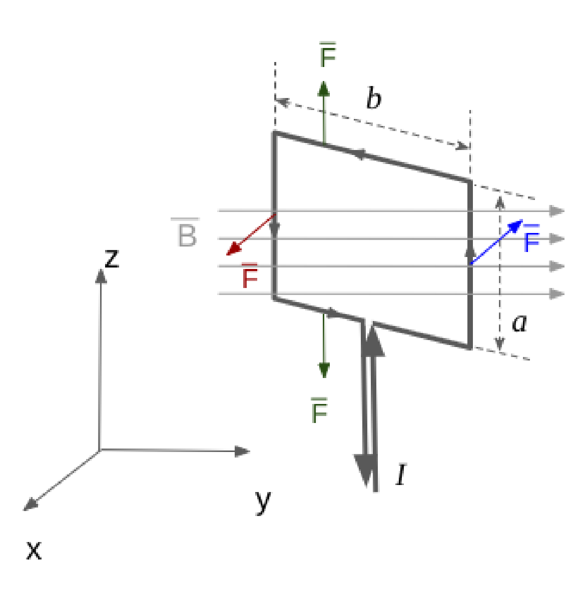


Figure 5.10. Magnetic force on a current loop placed in a way of a constant magnetic field.

$$d\overline{F}_1 = I\widehat{z} \times B\widehat{y} dz = - IBdz\,\widehat{x} \qquad (5.48)$$

This gives a force on the charge element that is point towards the negative x-axis as shown by the blue arrow in figure 5.10. The total force on the side is obtained by integrating over the length of the wire. When both the current and the magnetic fields are constant, the integration result is

$$\overline{F}_1 = - \int_{z=0}^{a} IBdz\,\widehat{x} = - IBa\widehat{x} \qquad (5.49)$$

Similarly the magnetic force over the other vertical side is

$$\overline{F}_3 = IBa\hat{x} \tag{5.50}$$

For the top and lower sides, the current is moving in the x-y plane. Hence, we can break it into two components: $\overline{I} = I_x\hat{x} + I_y\hat{y}$. The cross product term $\overline{I} \times \overline{B}$ becomes

$$\left(I_x\hat{x} + I_y\hat{y}\right) \times B\hat{y} = I_xB\hat{z} \tag{5.51}$$

The force on the upper side is pointing upwards along the z-direction. Similarly, the force on the lower side is pointing downwards.

Motion of the current loop under a constant field

The force on the upper and lower sides of the loop cancel each other and there should be no effect along the z axis. However, this is not the case for the vertical sides. The force on the right side in figure 5.10 is pushing it towards the negative x direction, while that on the left side is pulling it in the opposite direction. This will cause a rotation motion around the loop axis. As the loop rotates, the force component along the rotational direction, θ, reduces and it reaches zero when the loop is aligned along the x axis as illustrated in figure 5.11.

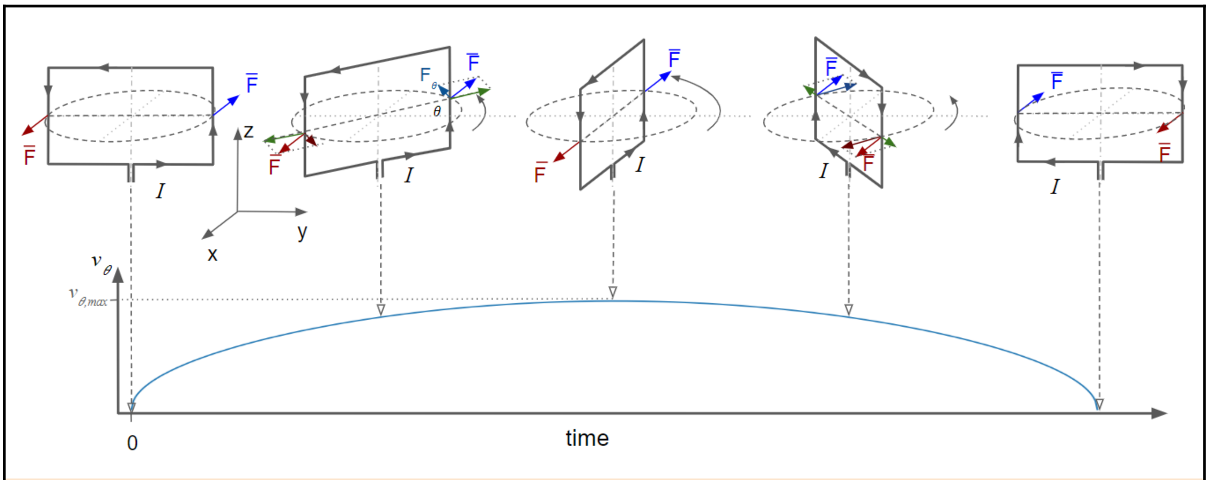


Figure 5.11. Motion of a current loop under a constant magnetic field in the y direction.

After that point, the force starts to resist the motion and the loop stops when it completes 180 degrees of rotation as shown by the right most diagram in figure 5.11. After this, the loop will switch direction and rotate in the opposite way stopping after 180 degrees and starts repeating the original cycle again. To find a mathematical representation of the motion, let us examine the rotation path at an intermediate time step as shown in figure 5.12. We know that the force on the right side is $\overline{F} =- IBa\hat{x}$. However, only the component along θ contributes to the rotational movement. The amplitude of this force is the projection of the force along the θ direction. From the geometry in figure 5.11, we can detect that $\hat{\theta} =- \cos\theta\,\hat{x} - \sin\theta\,\hat{y}$. Hence, the projection is

$$F_\theta = \left|\overline{F}\cdot\hat{\theta}\right| = IBa\cos\theta \tag{5.50}$$

If we assume that the side has a mass *M*, then from Newton's second law of motion we can detect that acceleration in the θ direction is

$\frac{dv_\theta}{dt} = F_\theta/m = IBa\cos\theta/M$ (5.51)

Where v_θ is the rotational speed. Let us consider that during an infinitesimal time dt the loop rotated by an angle $d\theta$. During the time, the side moves a distance $dl = v_\theta dt$ along the rotational path. We also know from the geometry that $dl = Rd\theta$.Hence we can write that

$d\theta/dt = v_\vartheta/R$ (5.52)

Applying this relation we can deduce the following relation

$\frac{dv_\theta}{dt} = \frac{dv_\theta}{d\theta}.\frac{d\theta}{dt} = v_\vartheta/R\frac{dv_\theta}{d\theta}$ (5.53)

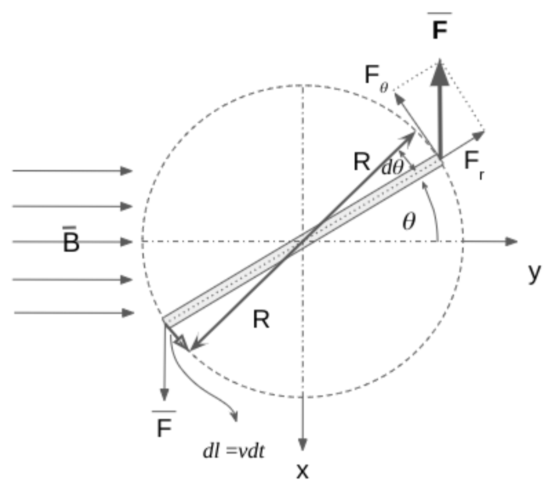


Figure 5.11. The path of the rotational movement of the current loop under a constant magnetic field.

We can use the relation in 5.53 in 5.51 as follows

$$v_\vartheta/R\frac{dv_\theta}{d\theta} = IBa\cos\theta/M \rightarrow v_\vartheta dv_\vartheta = IBRa\cos\theta/M\, d\theta \quad (5.54)$$

Performing integration over both sides we obtain

$$\frac{1}{2}v_\theta^2 = IBRa\sin\theta/M \rightarrow v_\vartheta = \sqrt{2IBRa\sin\theta/M} \quad (5.55)$$

From the geometry in figure 5.11 we know that $2R = b$, where b is the width of the loop. Hence,

$$v_\vartheta = \sqrt{B\,Iab\ \sin\theta/M} \quad (5.56)$$

The area of the loop $A = ab$ and the product IA is the magnitude of the magnetic moment of the current loop.

$$v_\vartheta = \sqrt{B\,m\ \sin\theta/M} \quad (5.57)$$

The maximum speed of rotation is $v_{\theta,max} = \sqrt{Bm/M}$ and it is achieved when the loop turns by 90º. The speed vanishes when the loop completes 180 degrees then starts rotating the opposite direction.

Torque on current loop

When considering a rotational movement, one important parameter is the torque. It represents the twist that causes the rotation. It can be written as the cross product of the force and the arm of rotation. In the geometry in figure 5.11, we can write the torque τ as

$$\overline{\tau} = \overline{F} \times \overline{R} = -IBa\,\hat{x} \times \overline{R} \quad (5.58)$$

The arm vector is

$$\overline{R} = \frac{b}{2}\left(-\ sin\theta\ \hat{x} +\ cos\theta\ \hat{y}\right) \quad (5.59)$$

The torque is then

$$\overline{\tau} =-\ \frac{1}{2} IBab\ cos\theta\ \hat{z} =-\ B\ m\ cos\theta\ \hat{z} \quad (5.60)$$

The toque is pointing along the negative z axis which is the axis of rotation.

One point to mention here is that in the motion we presented before, the loop was rotating back and forth. If however, we wish to make the loop continue rotating in the same direction, the current direction needs to flip right after the loop makes 180 degrees rotation. This can be achieved using a brush system as shown in figure 5.12. There, the two brushes are touching the wires, hence once the loop completes 180 degrees, the wires will switch the brush and the current direction return similar to the 0 degrees forcing the loop to continue rotating in the same direction.

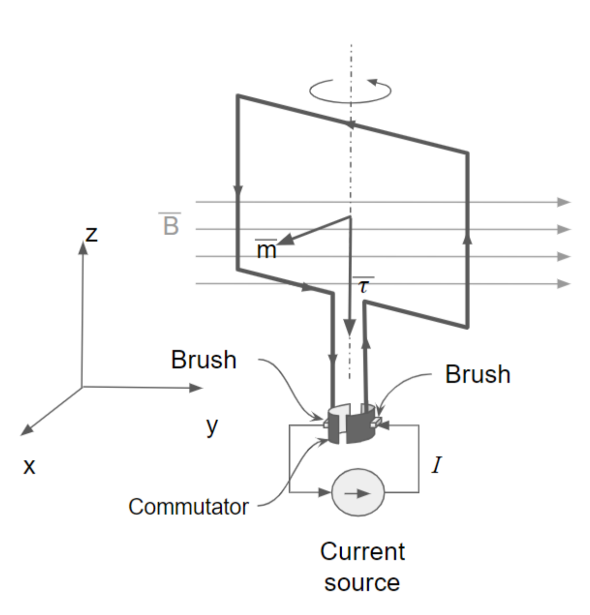


Figure 5.12. Adding conductor brushes to force the current loop to rotate along one direction.

Each end of the loop wire is connected to a commutator (a curved conducting surface) that is in touch with a conducting brush. The current is flowing in one direction. The gap between the commutators allows wire ends to switch the current directions when the loop completes 180 degrees of rotation. That allows the loop to continue rotating in the same direction. That is the main concept behind DC motors operation.