# Part: Electromagnetic radiation

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# Oscillating field

The charge acceleration a(t) and field E(t) can be calculated from position x(t)

$$x(t)=x_o\cos(\omega t)$$ Eq 2-1

$$a_x(t)=\frac{d^2x(t)}{dt^2}=-\omega^2 x_o\cos(\omega t)$$ Eq 2-2

Let $a_o=-\omega^2 x_o$, then

$$a_x(t)=a_o\cos(\omega t)$$ and Eq 2-3

$$E(t)=\frac{-q a_o}{4\pi\epsilon_o c^2 r}\cos(\omega(t-r/c))$$ Eq 2-4

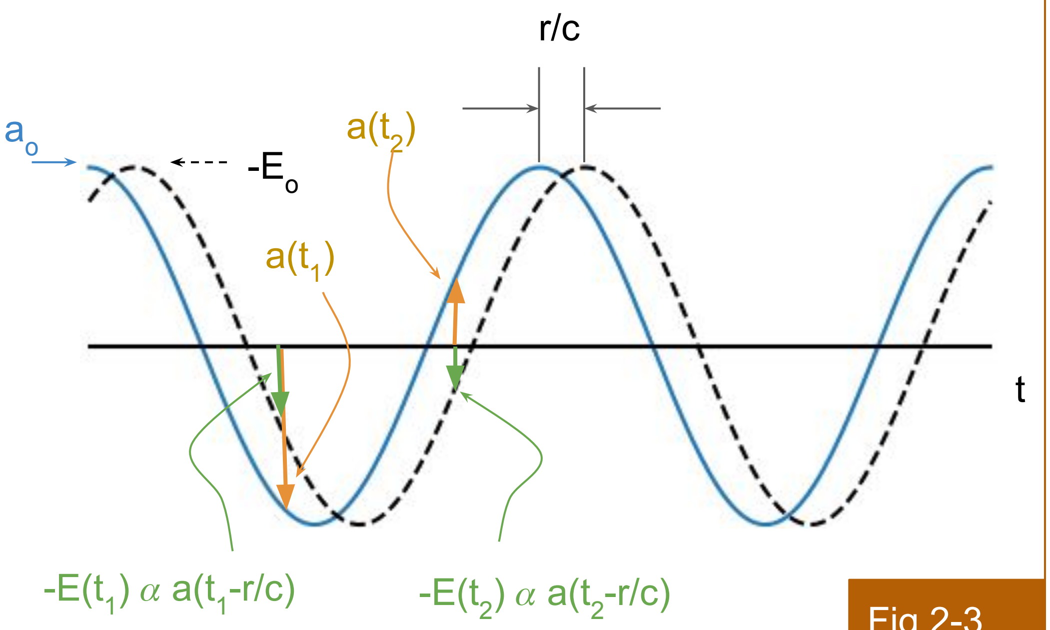


Fig 2-3

Let $E_o=\frac{-q a_o}{4\pi\epsilon_o c^2 r}$

$$E(t)=E_o\cos(\omega(t-r/c))$$ Eq 2-5

# Observer and radiation

In Eq 2-4 the observer was on the line of sight of the oscillating dipole. The acceleration vector was normal to the line of sight.

Now we move the observer with an angle θ.

In this case only the acceleration component normal to the line of sight contributes to the generation of the electric field.

$$E(t)=\frac{-q\,a_o \sin\theta}{4\pi\epsilon_o c^2 r}\cos\left(\omega\left(t-r/c\right)\right) \quad \text{Eq 2-6}$$

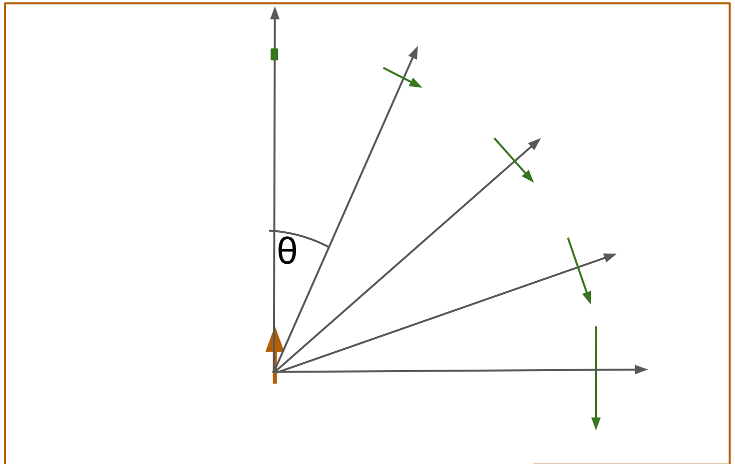


Fig 2-4

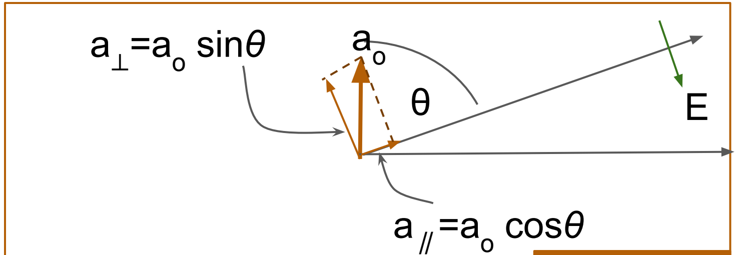


Fig 2-5

# Wave properties

Eq 2-6 can be written as an amplitude and phase

$$E(t)=E_o\cos(\phi) \text{ where } E_o=\frac{-q a_o \sin\theta}{4\pi\epsilon_o c^2 r} \text{ and } \phi=\omega(t-r/c) \qquad \text{Eq 2-7}$$

When we fix our observer at a point P and record the field variation over time, one notices that the field returns to its original state every time period $T=\frac{2\pi}{\omega}$ .

Similarly freezing time, E repeats itself every space period of $\lambda=\frac{2\pi c}{\omega}$, which is referred to as wavelength. The change of phase with respect to distance is $\frac{d\phi}{dr}=\omega/c=k$. The constant k is defined as the wave number with units of 1/m.

$$k=\omega/c=\frac{2\pi}{\lambda} \qquad \text{Eq 2-8}$$

# Field amplitude and force

The field amplitude is inversely proportional to distance from the dipole. At any instant of time if a particle of mass $m_p$ and charge $q_p$ is placed at one point in space, assuming linear response, it will move according to the force exerted on it. $\bar{F}(t)=m_p\bar{a}_p(t)=q_p\bar{E}(t)=q_pE_o\cos(\omega(t-r/c))\hat{e}_n$ Eq 2-9

Here $\mathbf{a_p}$ indicates the induced acceleration the particle gains by the force and $\mathbf{v_p}$ is the gained velocity.

$$\bar{a}_p(t)=\frac{q_pE_o}{m_p}\cos(\omega(t-r/c))\hat{e}_n$$ Eq 2-10

$$\bar{v}_p(t)=\frac{-q_pE_o}{\omega m_p}\sin(\omega(t-r/c))\hat{e}_n$$ Eq 2-11

# Energy transferred to a particle

The motion of the particle due to the electric field makes it gain a kinetic energy $T = \frac{1}{2} m_p v_p^2$.

$$T = \frac{q_p^2 E_o^2}{2\,\omega^2 m_p} \sin^2(\omega(t - r/c))$$ Eq 2-12

The energy gained by the particle reaches a maximum $T_{max}$ , decreases to zero and increases again over a period $\pi/\omega$.

$$T_{max} = \frac{q_p^2 E_o^2}{2\,\omega^2 m_p}$$ Eq 2-13

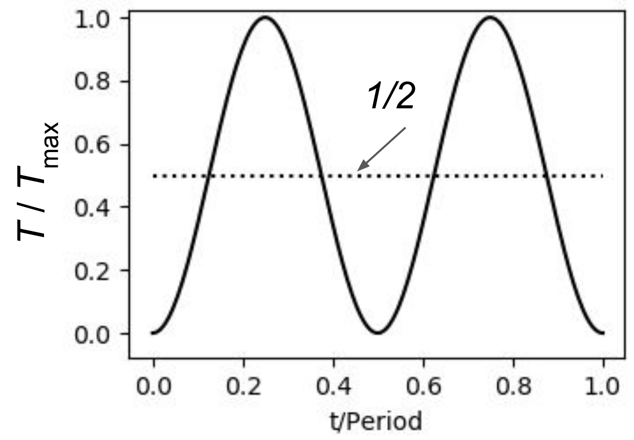


Fig 2-6

# Energy transferred to the particle

On average, the energy gained by the particle over a long period of time is then

$$\langle T\rangle=\langle\frac{q_b^2 E_o^2}{2\,\omega^2 m_p}\sin^2(\omega(t-r/c))\rangle=\frac{q_b^2 E_o^2}{2\,\omega^2 m_p}\langle\sin^2(\omega(t-r/c))\rangle \qquad \text{Eq 2-14}$$

The average of the square of a sinusoidal <sin$^2$a> = ½. Then

$$\langle T\rangle=\frac{q_p^2 E_o^2}{4\,\omega^2 m_p}=\frac{1}{2}T_{max} \qquad \text{Eq 2-15}$$

Here the particle gains on average a constant amount of kinetic energy that does not does not change over time. It is worth mentioning at this point that till now we have ignored the potential energy formed by the restoring force that pounds the charged particle to the molecule for example.

# Restoring force

One can think of the restoring force in a similar way that a spring works on an a mass attached to it. The force linearly depends on the position of the particle $\bar{F}_r = -k\bar{x}$ . Here k is a constant and the negative sign indicates a restoring force that points inwards. One can use the scalar form as the force and x are both along the x axis. The potential energy ,*U*, as the integration of F over a displacement distance x.

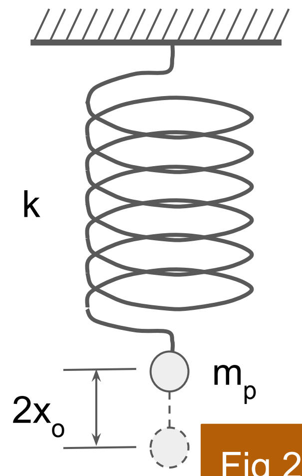


Fig 2-7

$$U = \int_0^x k\,x\,dx = \frac{1}{2} k\,x^2$$

Eq 2-16

$$= \frac{q_p^2 E_o^2 k}{2\,\omega^4 m_p^2} \cos^2(\omega(t - r/c))$$

Eq 2-17

# Stored energy

On average the potential energy is as well constant.

$$\langle U\rangle=\frac{q_p^2 E_o^2 k}{2\omega^4 m_p^2}\langle\cos^2(\omega(t-r/c))\rangle=\frac{q_p^2 E_o^2 k}{4\omega^4 m_p^2}$$ Eq 2-18

The total energy that is stored in the system on average is constant

$$\langle E\rangle=\langle U+T\rangle=\langle U\rangle+\langle T\rangle=const.$$ Eq 2-19

What about the power? The power in the system of oscillating particle is the rate of change of the work performed on the particle by the external force per second. Over an infinitesimal time dt the work done to displace the particle by dx is dW.

$$dW=\bar{F}\cdot\bar{dx} \rightarrow \frac{dW}{dt}=P=\bar{F}\cdot\frac{\bar{dx}}{dt}=\bar{F}\cdot\bar{v}_p$$ Eq 2-20

# Power in the system

The equation of motion the particle under the electric field ($F_{ex} = q_p E$)

$$\bar{F}(t) - k\bar{x} = m_p \frac{d^2\bar{x}}{dt^2} \rightarrow \bar{F}(t) = m_p \bar{a}(t) + k\bar{x}(t)$$

$$\bar{F}(t) = q_p E_o \left(1 - \frac{k}{\omega^2 m_p}\right) \cos(\omega(t - r/c))$$ Eq 2-21

The power is then

$$P(t) = \bar{F} \cdot \bar{v}_p = \frac{-q_p^2 E_o^2}{\omega m_p} \left(1 - \frac{k}{\omega^2 m_p}\right) \cos(\omega(t - r/c)) \sin(\omega(t - r/c))$$ Eq 2-22

The total power in the system when we average over a long time is zero

$$\langle P(t) \rangle = \frac{-q_p^2 E_o^2}{\omega m_p} \left(1 - \frac{k}{\omega^2 m_p}\right) \langle \cos(\omega(t - r/c)) \sin(\omega(t - r/c)) \rangle = 0$$ Eq 2-23

# Zero power

The result in Eq 2-23 is interesting. The average power in the system is zero. This is rather logical as on average the system keeps constant energy and hence the average time variation  is zero. We can derive this statement by observing that Eq 2-22 is nothing but the time derivative of the sum of kinetic and potential energies

$$P(t) = (2T_{max} - 2U_{max})\sin(\omega(t - r/c))\cos(\omega(t - r/c)) \quad \text{Eq 2-24}$$

$$= \frac{d}{dt}[T_{max}\sin^2(\omega(t - r/c)) + U_{max}\cos^2(\omega(t - r/c))] \quad \text{Eq 2-25}$$

$$= \frac{d}{dt}[T(t) + U(t)] \quad \text{Eq 2-26}$$

Having a zero power in the system is due to neglecting the damping factor in the oscillation.

# Damping oscillation

Let us now consider a damping force in the particle oscillation $\bar{F}_d = -m\gamma\bar{v}_p$. The negative sign indicate  resistance to the motion of the particle. One can think of it similar to a friction applied to a mass moving on a rough surface. In this case the equation of motion of the system is

$$\bar{F}(t) - m\gamma d\frac{\bar{x}}{dt} - k\bar{x} = m_p\frac{d^2\bar{x}}{dt^2} \rightarrow \bar{F}(t) = m_p\frac{d^2\bar{x}}{dt^2} + m\gamma d\frac{\bar{x}}{dt} + k\bar{x}$$ Eq 2-26

The power in the system is then

$$P(t) = \bar{F}\cdot\bar{v}_p = \frac{d}{dt}[U(t) + T(t)] + \gamma\frac{q_p^2 E_o^2}{\omega^2 m_p}\sin^2(\omega(t - r/c))$$ Eq 2-27

The average power is then only due to resistance

$$\langle P(t)\rangle = \gamma\frac{q_p^2 E_o^2}{2\omega^2 m_p}$$ Eq 2-28

# Natural frequency

Looking at Eq 2-25, the stored energy in the system at any time instance is

$$E(t) = T_{max} sin^2(\omega(t - r/c)) + U_{max} cos^2(\omega(t - r/c))$$ Eq 2-28

If there exists a frequency $\omega_o$ at which $T_{max}=U_{max}$, then stored energy becomes constant at any time.

$$\frac{q_p^2 E_o^2}{2\omega_o^2 m_p} = \frac{q_p^2 E_o^2 k}{2\omega_o^4 m_p^2} \rightarrow 1 = \frac{k}{\omega_o^2 m_p} \text{ or } \omega_o = \sqrt{\frac{k}{m_p}}$$ Eq 2-29

If we replace the frequency in Eq 2-21 by $\omega_o$ we obtain a zero force, and hence zero power. That gives a system that oscillates under zero external force and this frequency is referred to as natural frequency.

# Re-emission

When the charged particle move, bounded electron for example, it will generate another electrical field as well, $E_a$. Let us consider an observer at a distance behind the particle. The observer at point P’ feels the effect of all charges. Hence, it sees a system of two oscillating charges, the source and the particle at P. The total field at point P’ is then the superposition of these two fields

$$E_{p'} = E_a + E_s$$ Eq 2-30

One needs to keep in mind that there is no conservation of electric field.

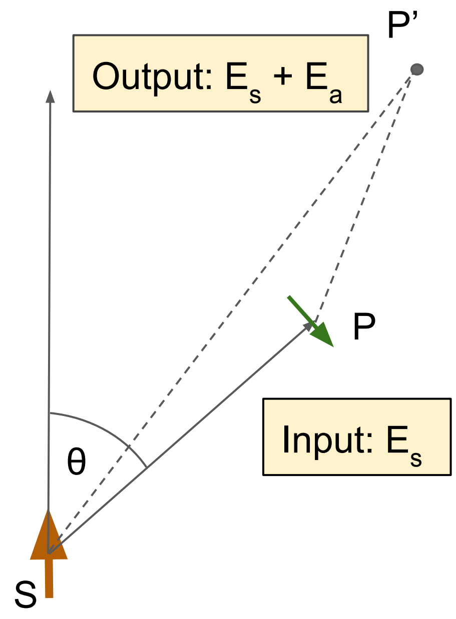


Fig 2-8

# Conservation of energy

For the oscillating system, the conservation of energy states that:
*Input energy per sec = output energy per sec + work done on particle per sec.*

From the previous discussion about the energy transfer to the particle one can predict that the output and emitted powers are proportional to the square of the field. For now we will set the proportionality constant could to be α.

$$\alpha E_s^2 = \alpha (E_s + E_a)^2 + F v_p \quad \text{Eq 2-31}$$

$$\alpha E_s^2 = \alpha (E_s^2 + E_a^2 + 2 E_s E_a) + q_p E_s v_p \quad \text{Eq 2-32}$$

Too small

The re-emitted field is considered to be much smaller than the source and hence one can neglect the square term of the field.

# Re-emitted field

Neglecting the $E_a^2$ term, Eq 2-32 is reduced to

$$0 = 2\alpha E_s E_a + q_p E_s v_p \rightarrow E_a = \frac{-q_p}{2\alpha} v_p$$ Eq 2-33

The re-emitted field is then proportional to the speed of the particle along the direction of oscillation. We will revisit this result later in part 5, however for now we need to finish the discussion of this part with the emitted power and the dependency on the distance. We will take the assumption used in deriving Eq 2-33 that the power proportional to the square of the field $\langle P_s \rangle = \alpha \langle E_s^2 \rangle$. The suffix s indicates the source. Using Eq 2-4, the average emitted power is

$$\langle P_s \rangle = \frac{1}{2} \alpha \left( \frac{q a_o}{4 \pi \epsilon_o c^2} \right)^2 \frac{\sin^2 \theta}{r^2}$$ Eq 2-34

# Power flow

The power in Eq 2-33 is measured at one particular point. If we look at an area dA on a spherical surface around point P. Every point on that area receives power $<P_s>$. Hence, the total amount of power radiated through the area should be proportional to dA. $<P_s>$ is proportional to $1/r^2$ while dA is proportional to $r^2$. Hence, the total power flowing through dA should not be dependant on the distance r.

$$dA = r^2 \sin\theta d\theta d\psi$$ Eq 2-35

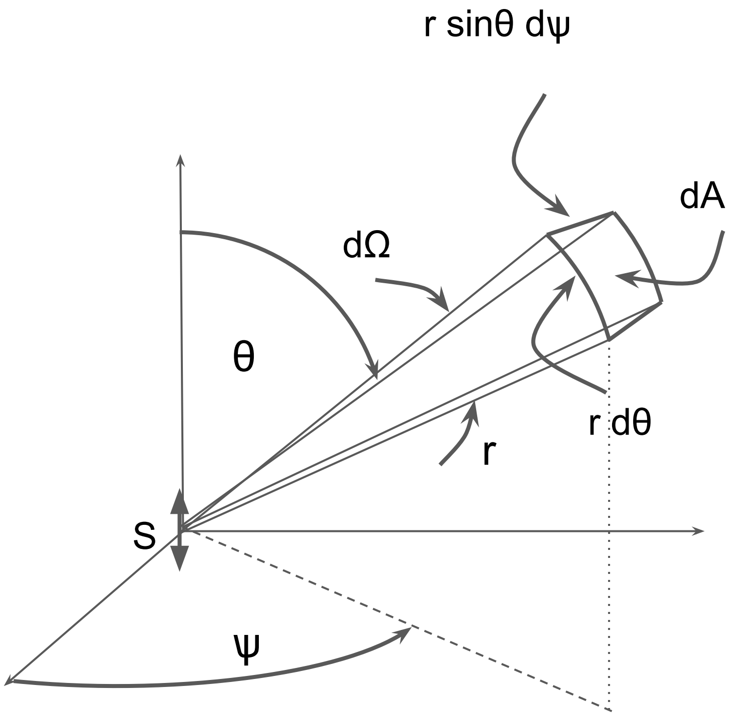


Fig 2-9

To mathematically represent this statement, let us assume there are M sample points within the area dA. The total power through the area dA is then

$$d\langle P_s\rangle = M\frac{1}{2}\alpha\left(\frac{qa_o}{4\pi\epsilon_o c^2}\right)^2\frac{sin^2\theta}{r^2} \qquad \text{Eq 2-36}$$

If we assume to have sampling density of η samples per unit area, then the number of samples M= ηdA.

$$d\langle P_s\rangle = \frac{\eta}{2}\alpha\left(\frac{qa_o}{4\pi\epsilon_o c^2}\right)^2\frac{sin^2\theta}{r^2}dA = P_o\frac{sin^2\theta}{r^2}dA \qquad \text{Eq 2-37}$$

Dividing Eq 2-37 by dA one obtains the expression for the intensity, power per unit area

$$\langle I_s\rangle = \frac{d\langle P_s\rangle}{dA} = P_o\frac{sin^2\theta}{r^2} \qquad \text{Eq 2-38}$$

# Radiated power

The intensity in Eq 2-38 is proportional to $1/r^2$. As for the power, if we substitute Eq 2-35 in Eq 2-37 one obtains the following

$$d\langle P_s\rangle = P_o sin^2\theta(sin\theta d\theta d\psi) = P_o sin^2\theta d\Omega \qquad \text{Eq 2-39}$$

The emitted power in a solid angle dΩ does not depend on the distance but mainly on the angle it makes to the dipole's axis, $\theta$. In Fig 2-10, equal powers flow through areas A1 and A2 as they form the same solid angle Ω around the same angle θ. The intensities however $\langle I_{s,2}\rangle/\langle I_{s,1}\rangle = r_1^2/r_2^2$.

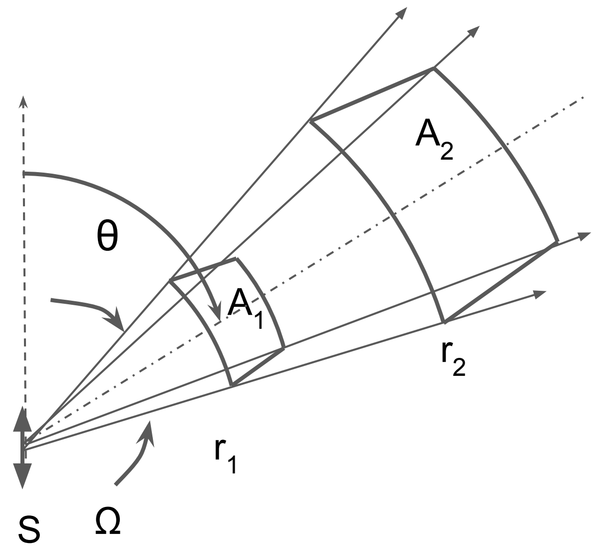


Fig 2-10

# Intensity as a function of field

Looking back at Eq 2-34, one can write a similar expression for the intensity from Eq 2-38 as follows

$$\langle I_s \rangle = \frac{1}{2}\eta\alpha\left(\frac{q a_o}{4\pi\epsilon_o c^2}\right)^2 \frac{\sin^2\theta}{r^2} \qquad \text{Eq 2-40}$$

The term ηq looks like a charge density (has units of $C/m^2$). It is like instead of looking at sample density we assume the charge q is distributed evenly on a surface with density ηq.

$$\langle I_s \rangle = \eta\alpha\langle E_s^2 \rangle \qquad \text{Eq 2-41}$$

Let us examine the units of the averaged terms in Eq 2-41. $<I_s>$ has units of $W/m^2$. $<E_s{}^2>$ has units of $V^2/m^2 = VA\Omega/m^2 = \Omega\ W/m^2$. Hence the term ηα has to have units of 1/Ω or conductance.

# Radiation damping

The Term $1/\eta\alpha$ has units of Ohm and it represents an impedance or in other words a free space resistance. In part 5, we will figure out that this resistance equals $1/\epsilon_o c$ or a value of 377 Ω.

As we discussed earlier, the power taken by any system requires a mean of loss in the system. That was represented by the damping oscillation in the spring model of the electron in a molecule. For radiation, the loss is then taken by the free space resistance. Hence, for the oscillating source dipole to keep radiating a constant power support is needed. The value of this power equals the integration of Eq 2-40 over a spherical surface area of radius r.

# Total radiating power

The total power taken through radiation and needs to be added to the oscillating source is then

$$\langle P_s \rangle = \frac{1}{2} \eta \alpha \left( \frac{q a_o}{4 \pi \epsilon_o c^2} \right)^2 \frac{4}{3} 2 \pi = \epsilon_o c \frac{q^2 a_o^2}{12 \pi \epsilon_o^2 c^4} = \frac{q^2 a_o^2}{12 \pi \epsilon_o c^3} \qquad \text{Eq 2-42}$$

Hence, this power is emitted into space and propagates till it gets absorbed by a charged particle.

Hence, an oscillating charge with no external effect will not oscillate forever, it constantly loses power as in Eq 2-42. That causes the damping in oscillation as will be discussed in more details in part six.

# Brief discussion

The field generated by oscillating dipole  has a phase, that is retarded by the ratio of the distance to the speed of light, and an amplitude that is inversely proportional to the distance.

The work produced on a uniformly distributed charges on any surface area covered by a solid angle Ω is independent of the distance from the dipole. It is however dependants on the inclination that solid angle make with dipole axis.

The intensity at any location P is inversely proportional to the square of the distance from the dipole.