Part 3: Interference

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Two oscillating dipoles

Let us consider two exact dipoles that are synchronized with no time delay. The dipoles are oscillating along the z-axis (normal to the screen). They are are separated by a distance d along the y-axis as in Fig 3-1.

An observer at point P_1 detects two electric fields each follow the dipoles oscillation with different time retardations r_1/c and r_1'/c, where r'=r+d.

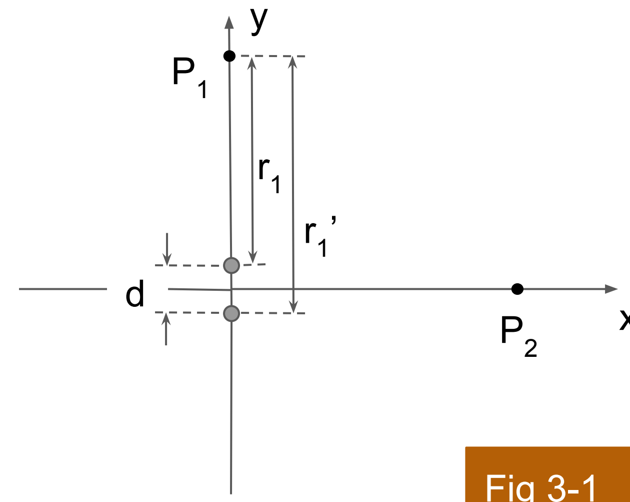


Fig 3-1

$$E(t) = \frac{V_o}{r} cos(\omega t - \omega r/c) \quad \text{Eq 3-1}$$

$$E'(t) = \frac{V_o}{r'} cos(\omega t - \omega r'/c) \quad \text{Eq 3-2}$$

$$V_o = E_o r = \frac{-q a_o \sin\theta}{4\pi\epsilon_o c^2}$$

In phase and out of phase

If the distances r and r' are much larger than d then the amplitudes V_o/r and V_o/r' can be assumed to be equal. The total field at point P1, which is a superposition of the two fields can be written as

$$E_1(t)=\frac{V_o}{r}\left[\cos\left(\omega t-\frac{\omega}{c}r\right)+\cos\left(\omega t-\frac{\omega}{c}r-\frac{\omega}{c}d\right)\right]_{\omega d/c=\pi}=0 \qquad \text{Eq 3-3}$$

If the term $\omega d/c = \pi$, or $d=\lambda/2$, Eq 3-3 gives a zero total field at point P_1. This is as the two waves are out of phase. If the distance d becomes λ (or an integer multiple of λ), the term $\omega d/c = 2\pi$. That makes the two terms between the brackets in Eq 3-3 the same and hence the total field amplitude is doubled. The total power is then four times that of a single dipole. The two waves are stated to be *in phase*.

Sources phase difference

An observer at point P_2 will find that both fields approaching the point are experiencing the same retardation ($r'_2=r_2$). Hence, regardless of the separation d, the two fields are always in phase as long as the two dipoles are synchronized without a delay.

If one of the dipoles is delayed such that it has a phase α then the total field is

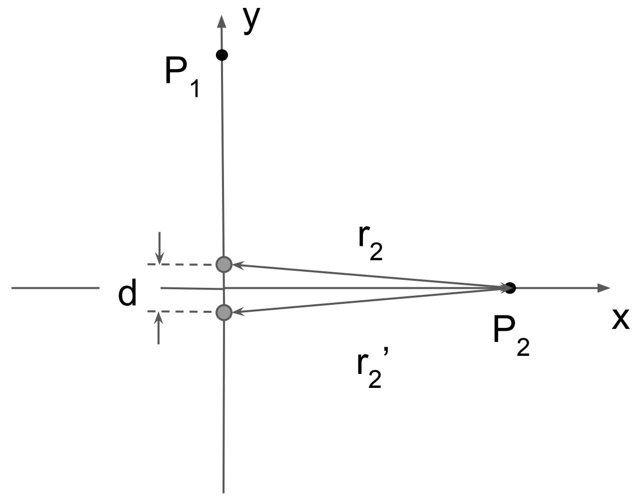


Fig 3-2

$$E_1(t)=\frac{V_o}{r}\left[\cos\left(\omega t-\frac{\omega}{c}r\right)+\cos\left(\omega t-\frac{\omega}{c}r-\alpha\right)\right]$$

Eq 3-4

Manipulation of radiation

Let us now set both dipoles to be in phase (α=0) and the spacing in between to be d=λ/2. In this case at P_1 the fields arrive with a π phase shift. The measured intensity is then equal to zero. This is the same for location P_3.

At P_2, the two fields come in phase and hence the measured intensity is 4 times that of a single dipole.

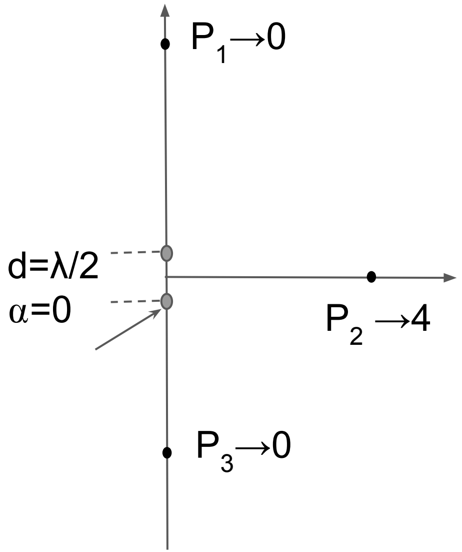


Fig 3-3a

Manipulation of radiation

If the phase of one dipole is set to be π relative to the other, then the total field at P_1 becomes in phase (total phase difference is then $\pi + \pi = 2\pi$). That gives a total intensity of 4 times that of one dipole. However, at point P_2 the two fields are out of phase now and the total field is zero.

Note that at point P_3 located the field, and power, is the same as P_1. The question is can we force the power to go on direction not the other?

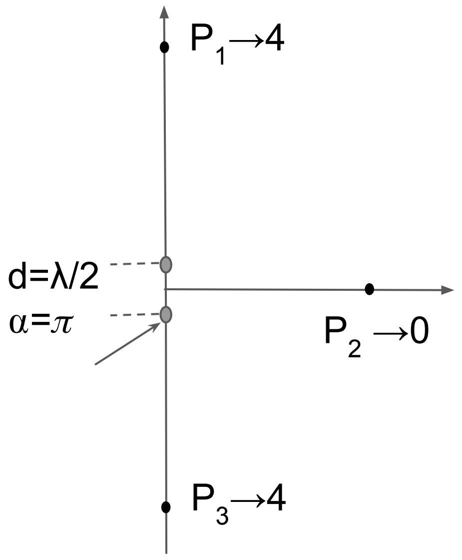


Fig 3-3

Directional radiation

Let us set the spacing between the dipoles to be λ/4 giving a phase retardation of $\pi/2$. Let us as well set the phase of the lower dipole α to be $\pi/2$. The phases of the two fields at point P_1 are

$$\phi_1 = \frac{\omega r}{c} \text{ and } \phi_1' = \frac{\omega r}{c} + \frac{\omega d}{c} + \alpha \qquad \text{Eq 3-5}$$

$$\phi_1' = \frac{\omega r}{c} + \frac{\pi}{2} + \frac{\pi}{2} = \frac{\omega r}{c} + \pi \qquad \text{Eq 3-6}$$

The two fields are out of phase and the intensity at P_1 is 0. At P_3 however the two fields are in phase.

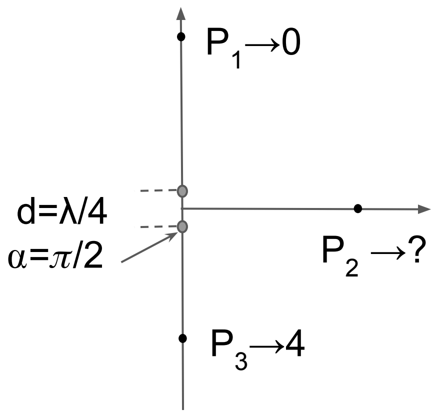


Fig 3-4

$$\phi_3 = \frac{\omega r}{c} + \frac{\omega d}{c} = \frac{\omega r}{c} + \frac{\pi}{2} \qquad \text{Eq 3-7}$$

$$\phi_3' = \frac{\omega r}{c} + \alpha = \frac{\omega r}{c} + \frac{\pi}{2} \qquad \text{Eq 3-8}$$

What about the field at point P_2?

Single dipole and double dipoles power

For now let us set the field amplitude to be E_o assuming all the detections are done at a distance r from the dipoles. Hence the total field by the two oscillators of equal amplitude is

$$E_{total}(t) = E_o[\cos(\omega t - \phi) + \cos(\omega t - \phi')]$$ Eq 3-9

Following the same approach as in Eq 2-41, the intensity is proportional to time average over $E_{total}(t)^2$. Now, for one dipole The intensity at a point P is

$$\langle I \rangle = I_o = \eta \alpha \langle E^2 \rangle = \epsilon_o c \langle E_o^2 \cos^2(\omega t - \phi) \rangle = \frac{1}{2} \epsilon_o c E_o^2$$ Eq 3-10

Here I_o is the intensity radiated by a single dipole.

Interference and intensity

For the double dipoles one can write the following equation for the intensity

$$\langle I\rangle = I_{total} = \epsilon_o c \langle E_o^2 [\cos(\omega t - \phi) + \cos(\omega t - \phi')]^2 \rangle$$ Eq 3-11

Using Eq 3-10 and expanding the square

$$I_{total} = \epsilon_o c E_o^2 \langle [\cos^2(\omega t - \phi) + \cos(\omega t - \phi')^2 + 2\cos(\omega t - \phi)\cos(\omega t - \phi')] \rangle$$

$$= 2 I_o [\frac{1}{2} + \frac{1}{2} + \langle \cos(2\omega t - \phi - \phi') + \cos(\phi - \phi') \rangle]$$ Eq 3-12

Time average for terms function of t is zero. That simplifies Eq 3-12 to

Eq 3-13

$$I_{total} = 2 I_o [1 + \cos(\phi - \phi')]$$

When ϕ-ϕ'=2mπ, the intensity is four time that of a single dipole. If it equals (2m+1)π, there will be no intensity observed at the point of detection.

Interference

In general, for equal amplitudes the intensiy at any point is

$$I_{total} = 2 I_o [1 + \cos(\phi - \phi')]$$ Eq 3-14

Now back to the power at P_2

$$\phi = \frac{\omega}{c} r \text{ and } \phi' = \frac{\omega}{c} r + \alpha$$

$$\phi' - \phi = \alpha = \frac{\pi}{2}$$ Eq 3-15

Substituting into Eq 3-14 $I_{total} = 2 I_o [1 + \cos(\pi/2)] = 2 I_o$ Eq 3-16

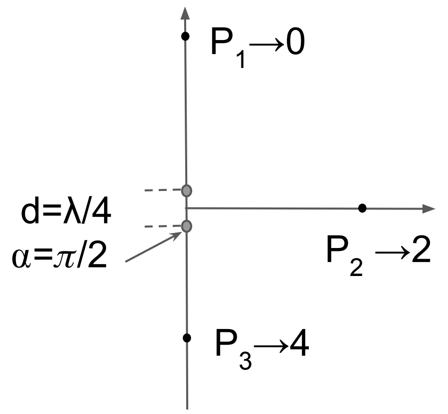
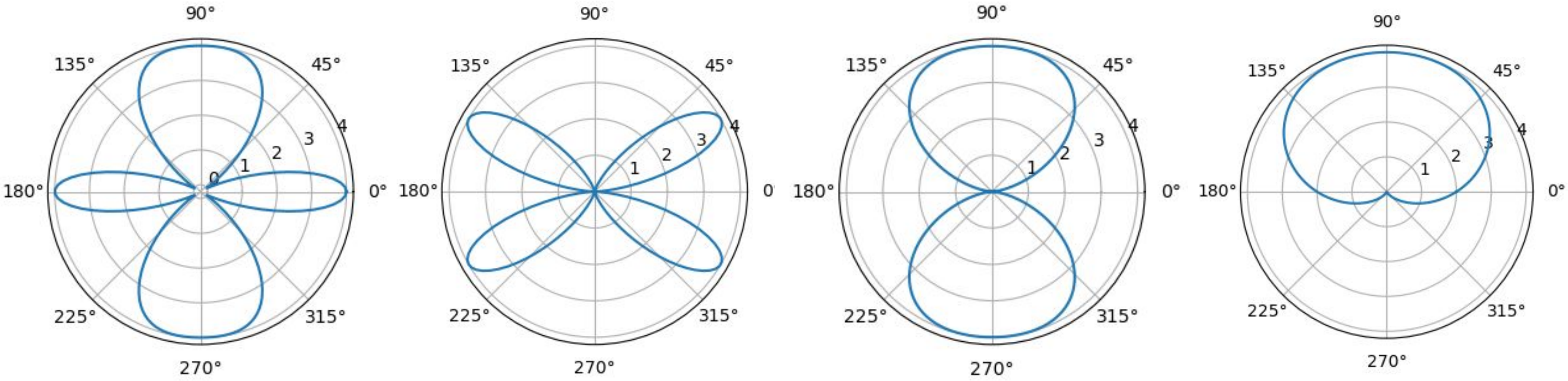


Fig 3-5

Controlling two-dipoles radiation pattern

Below are some radiation patterns for different phase difference and separation



$d = \lambda$	$d = \lambda$	$d = \lambda/2$	$d = \lambda/2$
$\alpha = 0$	$\alpha = \pi$	$\alpha = \pi$	$\alpha = \pi/2$

Fig 3-6

Practical aspects

How can we make make such dipoles? One way to think about making one oscillating dipole is to use a generator and two wires W_a and W_b antenna as in figure 3-6. The generator produces alternating potential with a period of $T=2\pi/\omega$.

The configuration in Fig 3-3 For example requires two of such antennas placed in parallel. For microwave (frequency of GHz or 10^9 Hz), the spacing between the dipole is in the range of 1.5 cm's for 10 GHz frequency for example.

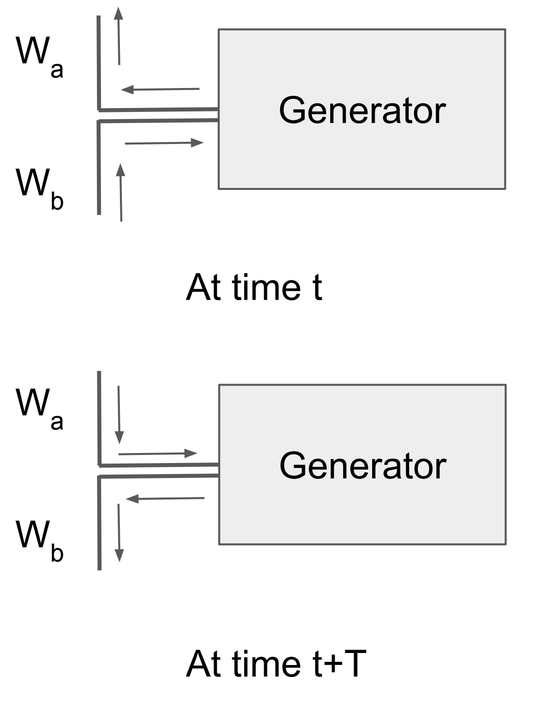


Fig 3-7

More dipoles

Let us consider three dipoles. The total field at any point is a superposition of the three fields each retarded by the distance over the speed of light. Following the same approach as Eq 3-13, the intensity is the time average of the square of the fields superposition.

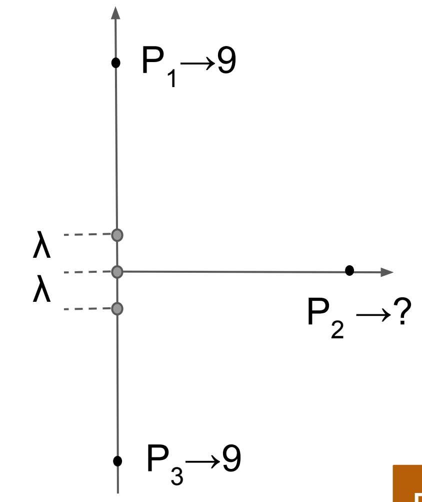


Fig 3-8

$$I_{total}=2I_o\langle[\cos(\omega t-\phi_1)+\cos(\omega t-\phi_2)\cos(\omega t-\phi_3)]^2\rangle$$ Eq 3-17

For points P_1 and P_3 in the configuration in Fig 3-5, the three fields are in phase. Hence

$$I_1=I_3=2I_o\langle[3\cos(\omega t-\phi_1)]^2\rangle=9I_o$$ Eq 3-18

Intensity pattern

We can work out Eq 3-18 to find a form similar to Eq 3-14

$$I_{total} = 2I_o[3 + 2\cos(\phi_1 - \phi_2) + 2\cos(\phi_2 - \phi_3) + 2\cos(\phi_3 - \phi_1)]$$ Eq 3-19

The radiation pattern has an interesting side loop between each major loop. This is clearly observed in The linear plot in Fig 3-9.

What about more dipoles?

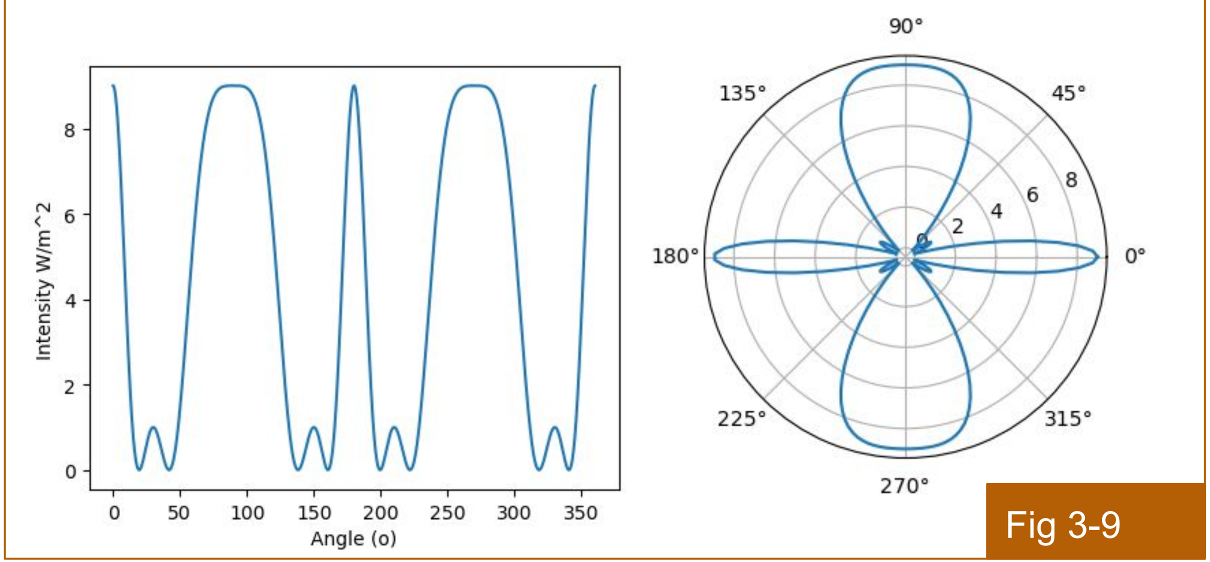


Fig 3-9

Multiple dipoles

If there are more dipoles to be considered then finding a form similar to Eq 3-19 becomes quite hard. One way to solve this issue is to recall that a cosine can be written as a sum of two complex values $\cos(\omega t - kr) = \frac{e^{i(\omega t - kr)} + e^{-i(\omega t - kr)}}{2}$

The statistical average over time

$$\langle \cos(\omega t - kr)^2 \rangle = \frac{1}{4} \langle e^{i2(\omega t - kr)} + e^{-i2(\omega t - kr)} + 2 e^{i(\omega t - kr)} e^{-i(\omega t - kr)} \rangle$$

$$= \frac{1}{4} \langle e^{i2(\omega t - kr)} \rangle + \frac{1}{4} \langle e^{-i2(\omega t - kr)} \rangle + \frac{1}{2} \langle e^{i(\omega t - kr)} e^{-i(\omega t - kr)} \rangle \qquad \text{Eq 3-20}$$

Notice that the two first terms average out to zero and the last term gives us the expected result.

Complex notation

In the last term of Eq 3-20, one can remove ωt and write the equation as

$$\langle \cos(\omega t - kr)^2 \rangle = \frac{1}{2} \langle e^{ikr} e^{-ikr} \rangle = \frac{1}{2} e^{ikr} e^{-ikr} \qquad \text{Eq 3-21}$$

The time average is then equivalent to multiplying the real part of the time independent term of the cosine by its complex conjugate and divide by 2. Hence Eq 3-17 and 3-19 could have been written differently as

$$I_{total} = I_o \left(e^{i\phi_1} + e^{i\phi_2} + e^{i\phi_3}\right)\left(e^{-i\phi_1} + e^{-i\phi_2} + e^{-i\phi_3}\right) \qquad \text{Eq 3-22}$$

$$= I_o \left|e^{i\phi_1} + e^{i\phi_2} + e^{i\phi_3}\right|^2 \qquad \text{Eq 3-23}$$

Multiple dipoles

Now we shall be able to deal with N dipoles each having a phase $\phi_h = kr_h + \alpha_h$

$$I_{total} = I_o \left| \sum_{h=1}^{N} e^{i\phi_h} \right|^2$$ Eq 3-23

If the dipoles generate different field amplitudes then Eq 3-23 can be written as

$$I_{total} = \epsilon_o c \left| \sum_{h=1}^{N} E_{o,h}^2 e^{i\phi_h} \right|^2$$ Eq 3-24

Where α is defined in Eq 3-10

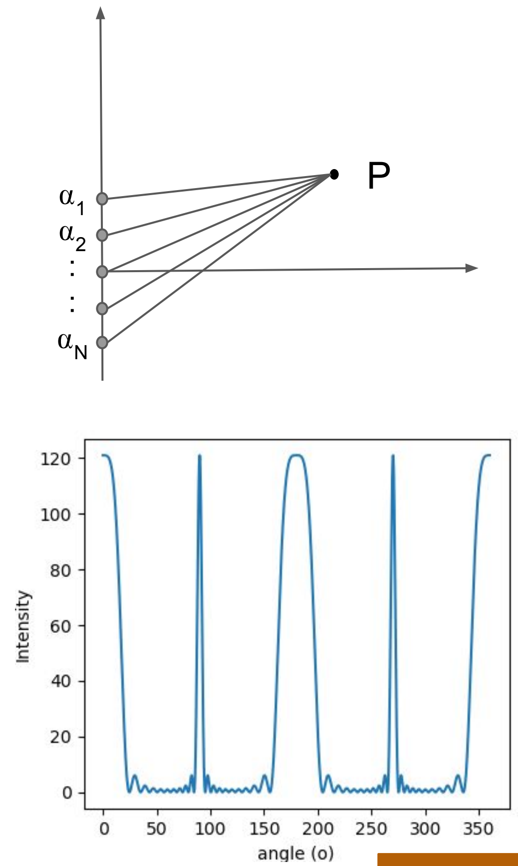


Fig 3-10

Non-interfering dipoles

Let us look back into two dipoles interference in Eq 3-11. If we now consider that each dipole is driven separately. To say, the two dipoles are not wired together. In this case, it becomes extremely hard to guarantee that both dipoles will oscillate in exactly the same frequency. In this case Eq 3-11 becomes

$$\begin{aligned}\langle I\rangle = I_{total} &= \epsilon_o c \langle E_o^2 [\cos(\omega t - \phi) + \cos(\omega' t - \phi')]^2 \rangle \\ &= 2 I_o [1 + \langle \cos((\omega - \omega')t - (\phi - \phi')) \rangle] = 2 I_o\end{aligned} \qquad \text{Eq 3-25}$$

In this case, cross term in Eq 3-25 becomes time dependant and the time average is zero. Hence, the total intensity becomes the sum of each individual dipole intensities.

Coherence

Till now we talked about two cases, one that interference (100%) and one that does not interfere (0%). The question is, is it always the case? Actually one can predict that there are more gray shades in between. This is the case when we have “some” interference (<100% and >0%).

If the two dipoles are perfectly synchronized, one can say that the fields emitted by the two topiles are coherent and hence we see interference.

If we can not synchronize them, which is case we just studied, then they are not coherent or incoherent.

If it is in between the two limits, we can them to be partially coherent.

Coherence representation

To be able to represent coherence, one can assume that the field emitted by any dipole is “more or less” oscillating with intended frequency. However, its amplitude might change slightly with time.

$$E_{total}(t) = E_1\left(t - \frac{r}{c}\right)\cos(\omega t + \phi) + E_2\left(t - \frac{r'}{c}\right)\cos(\omega t + \phi') \qquad \text{Eq 3-26}$$

The recorded intensity in this case is

$$\langle I \rangle = \varepsilon_o c \langle E_{total}^2 \rangle = \varepsilon_o c \left\langle \left(E_1\left(t - \frac{r}{c}\right)\cos(\omega t + \phi) + E_2\left(t - \frac{r'}{c}\right)\cos(\omega t + \phi') \right)^2 \right\rangle \qquad \text{Eq 3-27}$$

Expanding the square term and applying the average of cosines as before,

Coherence and interference

$$\langle I \rangle = \frac{1}{2}\varepsilon_o c \langle E_1^2\left(t - \frac{r}{c}\right)\rangle + \frac{1}{2}\varepsilon_o c \langle E_2^2\left(t - \frac{r'}{c}\right)\rangle$$
$$+ \varepsilon_o c \langle E_1\left(t - \frac{r}{c}\right) E_2\left(t - \frac{r'}{c}\right)\cos(\omega t + \phi)\cos(\omega t + \phi')\rangle$$

Eq 3-28

The first two terms are the intensities measured at the observation point produced by each dipole individually.

$$\langle I \rangle = \underbrace{\langle I_1 \rangle + \langle I_2 \rangle}_{\text{Individual intensities}} + \underbrace{\varepsilon_o c \langle E_1\left(t - \frac{r}{c}\right) E_2\left(t - \frac{r'}{c}\right)\rangle \cos(\phi - \phi')}_{\text{Interference term}}$$

Eq 3-29

Looking now at the interference term

Coherence coefficient

$$\varepsilon_o c \left\langle E_1\left(t-\frac{r}{c}\right) E_2\left(t-\frac{r'}{c}\right)\right\rangle = \varepsilon_o c \int_t E_1\left(t-\frac{r}{c}\right) E_2\left(t-\frac{r'}{c}\right) dt$$ Eq 3-30

Defining t’ = t-r/c and *τ* =r/c - r’/c,

$$\Gamma(\tau) = \varepsilon_o c \int_{t'} E_1(t')E_2(t'-\tau)\, dt'$$ Eq 3-31

Using Eq 3-31, we can write Eq 3-29 as

$$\langle I\rangle = \langle I_1\rangle + \langle I_2\rangle + \Gamma(\tau)\cos(\phi - \phi')$$

The term $\Gamma(\tau)$ is defined as the coherence coefficient

Types of sources

Incoherent source, no interference

Partial coherent source, fringes are within specific delay

Coherent source, where interference occurs regardless of delay.