Part 4: Diffraction

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Dipoles at far distance

Let us move the observer in FIg 3-10 to a quite far distance such that all the lines emitting from the dipoles are approximately parallel. If the two dipoles were synchronized with no delay, the observer at angle θ finds that one wave is delayed compared to the other such that the phase difference between the two waves is. $\Delta\phi = \frac{2\pi}{\lambda} d \sin\theta$. The two waves are in phase when the phase difference is multiple of 2π.

$$\frac{2\pi}{\lambda} d \sin\theta = 2m\pi \rightarrow \sin\theta = m\frac{\lambda}{d} \qquad \text{Eq 4-1}$$

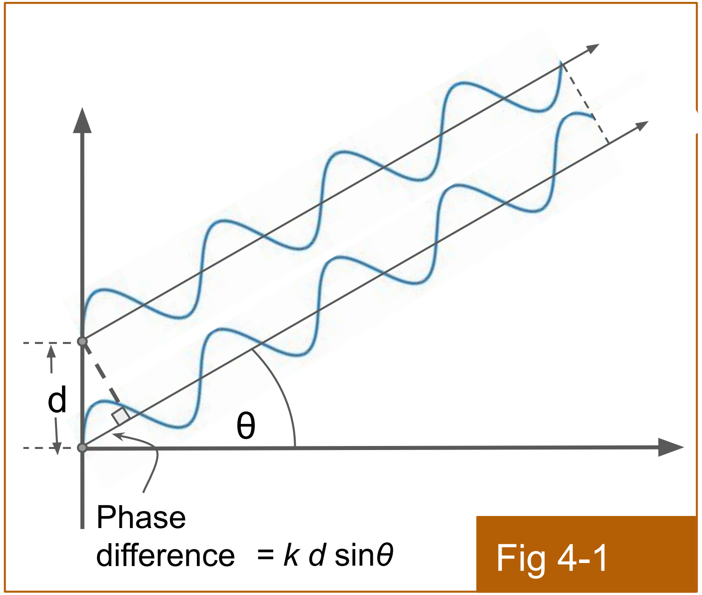


Fig 4-1

Radiation pattern at far observer

Let us place a sphere of a large radius with the two oscillators centered. The intensity pattern observed show distinctive pattern of peaks and valleys (Fringes). When d=2λ, Eq 4-1 gives us three possible integer values of m such that $\sin\theta$ is less than or equal to one: m=0,1 and 2.

If we increase the spacing between the dipoles to 5λ for example there will exist six values for which the sinusoidal term is smaller or equal to 1, namely m=0,1,2,3,4 and 5.

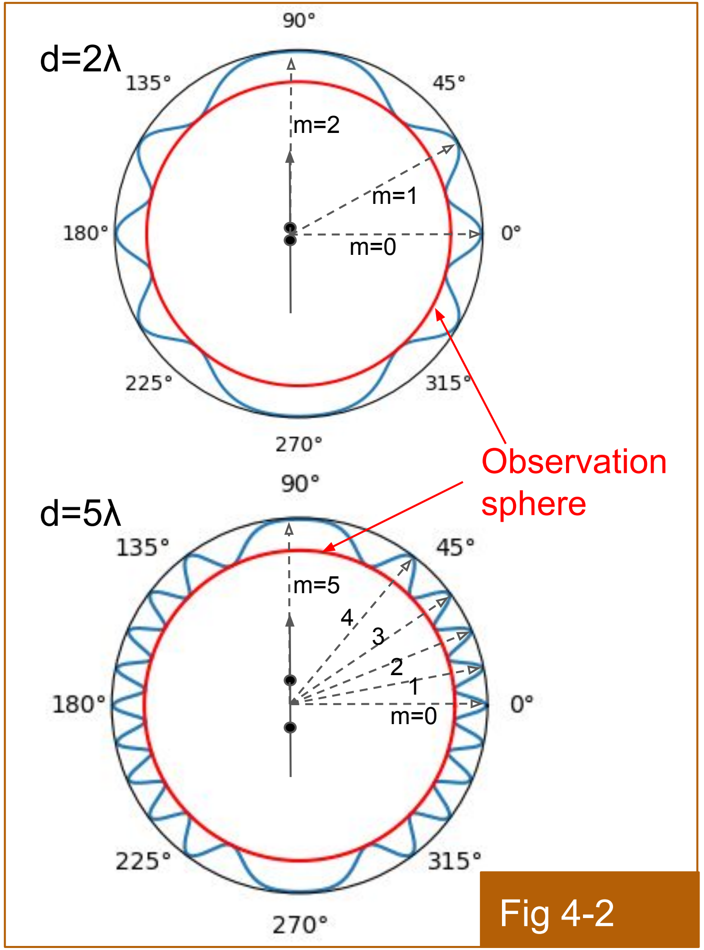


Fig 4-2

Filling the gap with dipoles

Now let us fill the gap between the two dipoles with N -1 identical dipoles which are all synchronized. The intensity pattern can be written as in Eq 3-23

$$I_{total} = I_o \left| \sum_{n=1}^{N} e^{i\phi_o + n\Delta\phi} \right|^2, \text{ where } \Delta\phi = \frac{2\pi}{\lambda} \delta \sin\theta \quad \text{Eq 4-2}$$

We can take the term ϕ_o out and look at the summation

$$A = \sum_{n=0}^{N} e^{i(n\Delta\phi)} = ?? \quad \text{Eq 4-3}$$

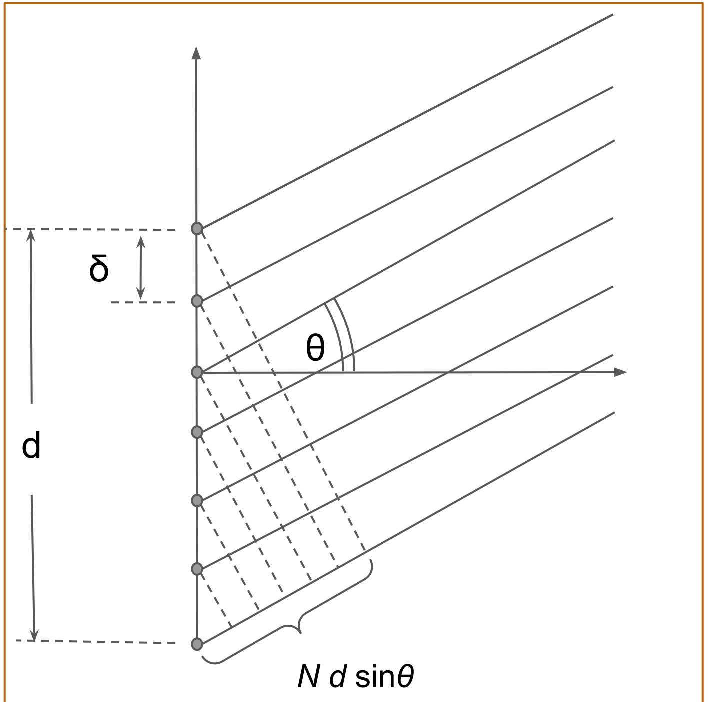


Fig 4-3

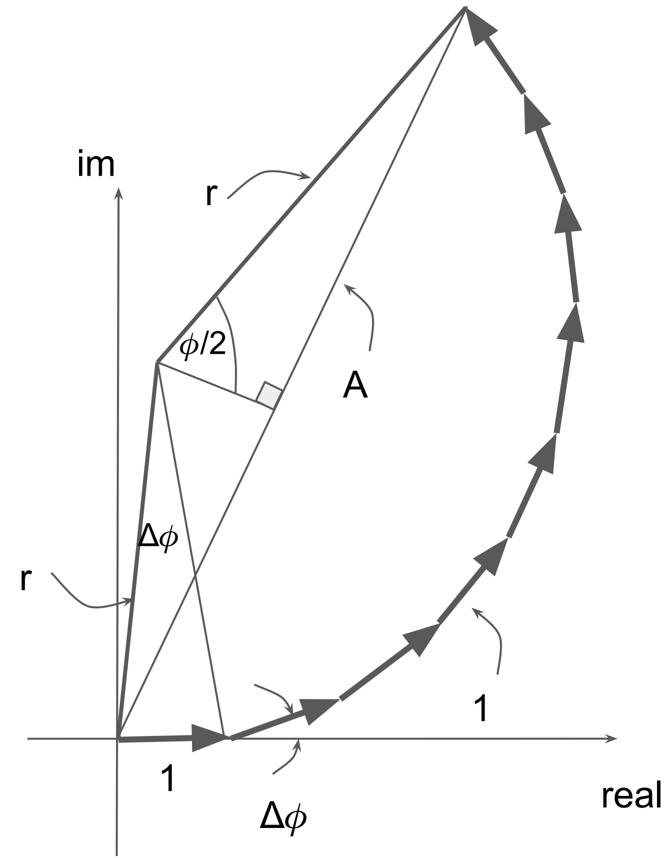
Many dipoles

Feynman in his lectures on physics presented a geometrical solution to find a closed form for Eq 4-3 as a summation of different vectors each has amplitude 1 and phase $\Delta\phi$ as in Fig 4-4. The amplitude of the summation A is

$$A=\sum_{n=0}^{N} e^{i(n\Delta\phi)}=2r\sin\left(N\frac{\Delta\phi}{2}\right) \qquad \text{Eq 4-4}$$

From the figure, $1=2r\sin\left(\frac{\Delta\phi}{2}\right)\rightarrow r=\frac{1}{2\sin\left(\frac{\Delta\phi}{2}\right)}$

$$A=\frac{\sin\left(N\frac{\Delta\phi}{2}\right)}{\sin\left(\frac{\Delta\phi}{2}\right)} \qquad \text{Eq 4-5}$$



$$\phi=\frac{2\pi}{\lambda}N\,\delta\sin\theta=\frac{2\pi}{\lambda}d\sin\theta$$

Fig 4-4

Limits

Now let us take the limits when the number of dipoles becomes very large and the spacing between them,δ, becomes infinitesimally small. The sinusoidal term at the denominator of Eq 4-5 can be approximated to equal the phase $\Delta\phi$.

$$A \approx \frac{\sin(\phi/2)}{\Delta\phi/2} = N\frac{\sin(\phi/2)}{\phi/2}$$ Eq 4-6

$$I_{total} = I_N \frac{\sin^2(\frac{\phi}{2})}{(\frac{\phi}{2})^2}$$ Eq 4-7

Where $I_N = N^2 I_o$ and I_o is from individual dipoles.

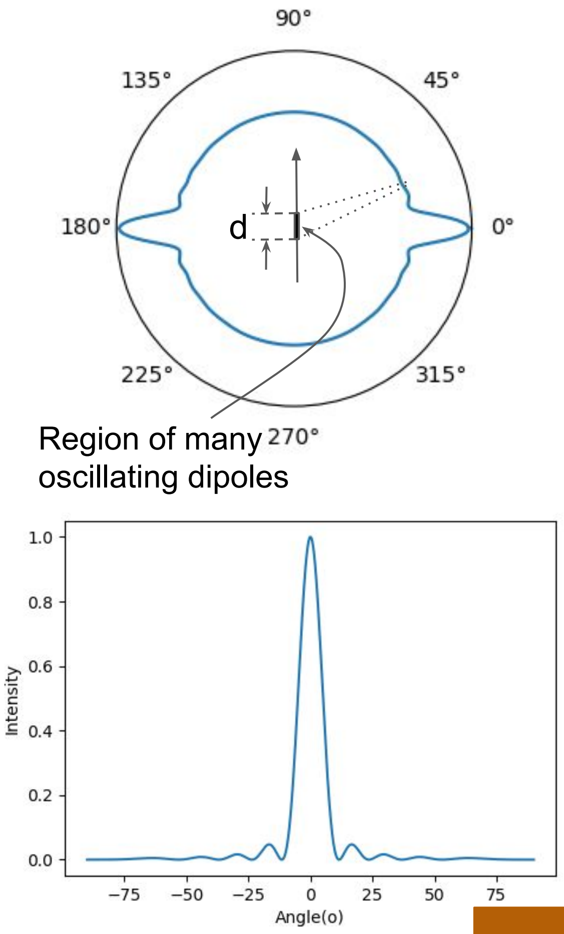


Fig 4-5

Diffraction grating

Now let us look at the configuration in Fig 4-3 differently. Practically, it is hard to generate many dipoles and connect them to the same source keeping no delay between any two dipoles. It is possible but extremely hard. A more practical way is to use an external source that generates an oscillating field. The source is placed far away such that each dipole sees the same delay at the "region of many dipoles," which we will call from now on diffraction grating.

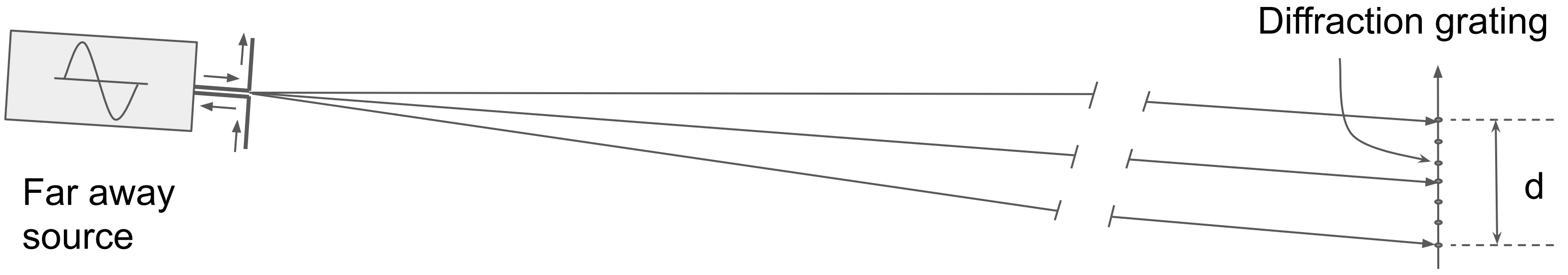


Fig 4-6

Dipoles re-radiation

The incident electric field exerts a force on each dipole causing them to oscillate. Each dipole emits a field as discussed in the earlier sections. The far field intensity can be calculated from Eqs 4-4 and 4-5. However, the phase difference between each dipole will be affected by the angle the incident field makes with the grating. If we observe the top two dipoles. The field reaches the second dipole with a delay δ sinθi. When rediating, the emitted field experience an extra delay of δ sinθ_d. The total phase difference between any two dipoles is then →

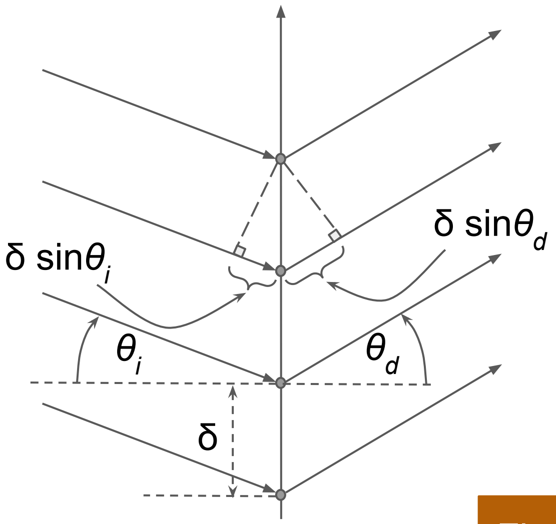


Fig 4-7

$$\Delta\phi = \frac{2\pi}{\lambda}\delta sin\theta_i + \frac{2\pi}{\lambda}\delta sin\theta_d$$

Eq 4-8

Diffraction equation

In order to achieve a peak intensity (constructive interference), the phase difference between two consecutive dipoles need to be multiple of 2π.

$$\sin\theta_i + \sin\theta_d = \frac{m\lambda}{\delta}$$ Eq 4-9

Eq 4-9 is known as the diffraction equation. Notice that if we consider a notation where positive angles are measured clockwise, then in Fig 4-7 θ_i and θ_d have different signs. This is clear from Eq 4-9 when m=0, $\sin\theta_i = -\sin\theta_d$.

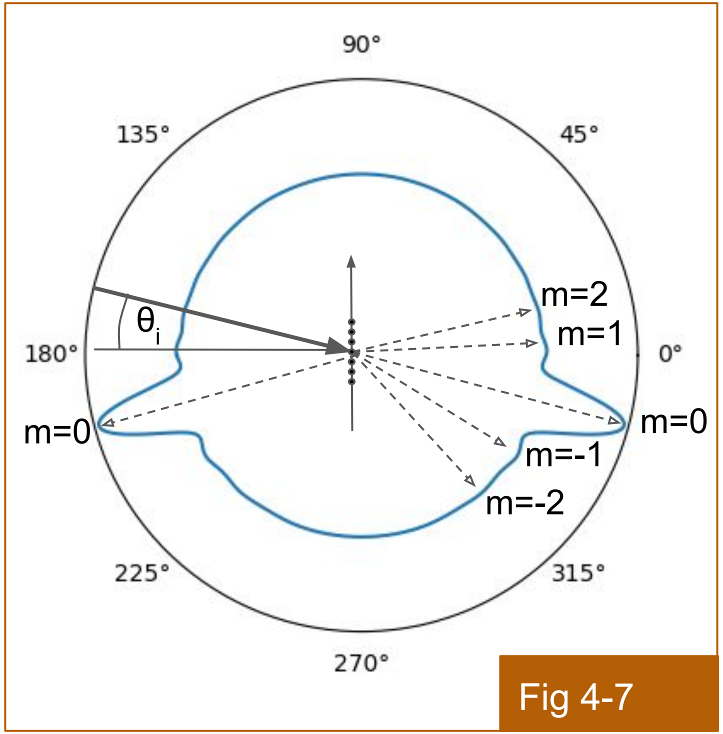


Fig 4-7

Angle notation

One can observe from Fig 4-8 that angles θ_i and θ_d are measured in two different directions. Hence, one needs to set a fixed notation. Hence, if positive is set to be clockwise then angle is θ_d negative. That explains the negative sign in the last result. If one replaces by θ_d in $-\theta_d$ Eq 4-9 we obtain the know diffraction equation

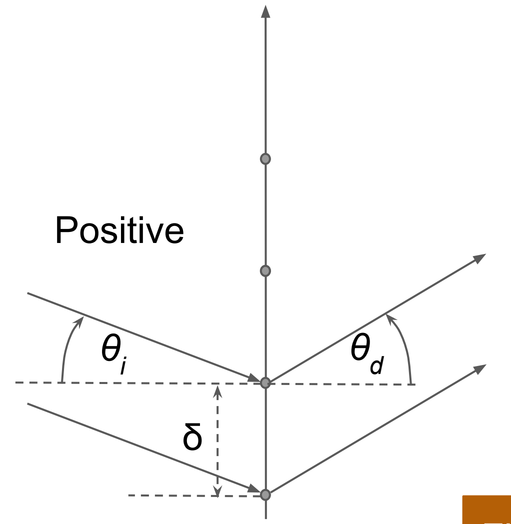


Fig 4-8

Diffraction by aperture

Now, let us visit of the region of large number of dipoles in Fig 4-5. But now we consider a case of an aperture of width d and a light shining on it. Would the light simply cast the shadow of the aperture where no light exists in the opaque region? The experimental observations show different response. It shows a clear overshooting near the edge as shown in Fig 4-9b. The question here is how to use the dipoles model to explain this response?

The answer is consider that the aperture is filled uniformly with dipoles instead of being the only place with no charge. But how?

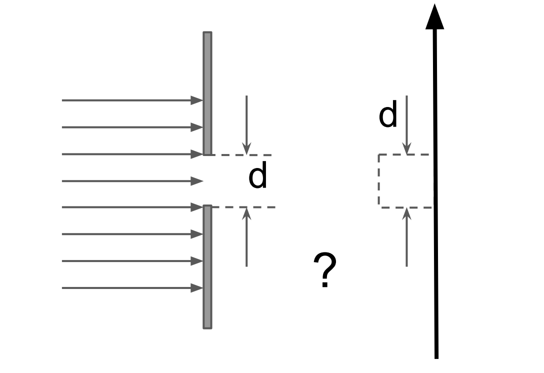


Fig 4-9a

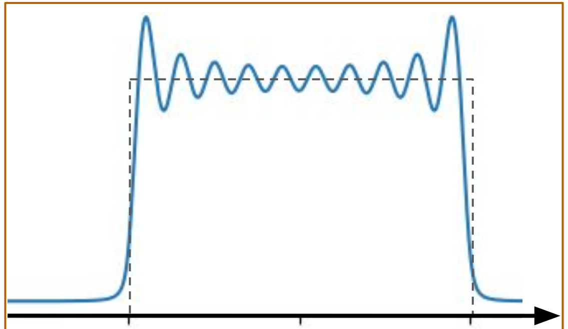


Fig 4-9b

Charges and fields

In order to understand the previous statement, consider the following scenario.

- A source is located far away in front of a very thin plate such that the field at all the charges on the plate has approximately the same retardation.
- The incident field (E_s) exerts a force on the charges causing them to accelerate accordingly forming oscillating dipoles.
- The dipoles emit electric fields in both directions: forward direction (E_a) and backward direction (E_r).

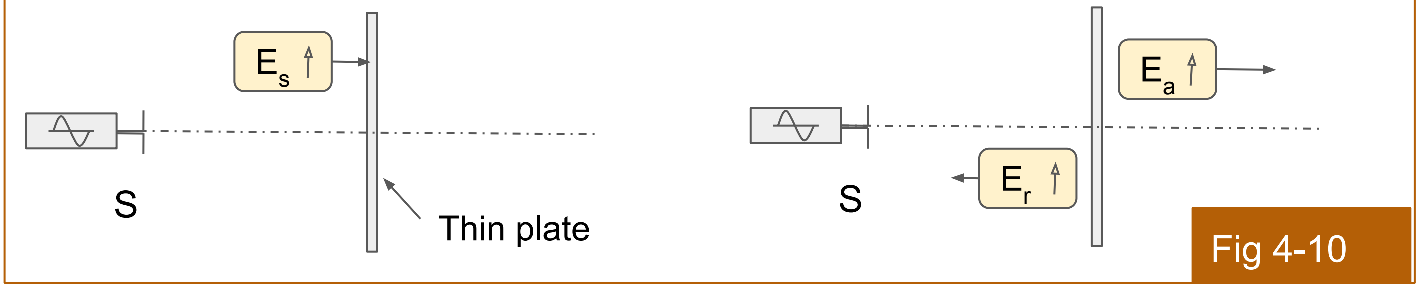


Fig 4-10

Field as a result of all possible charges

Now let us recall the following physical assumption:

The field at any point of space is the sum of all fields produced by all the charges in the universe.

So, for a point P located far behind the plate, the electric field is the sum of the one due to the source and the ones generated by all charges on the plate.

$$E_P = E_s + E_a \quad \text{Eq 4-10}$$

$$E_a = \sum E_{\text{plate charges}} \quad \text{Eq 4-11}$$

Here we neglected all other charges further away from the system and consider their effect to be negligible.

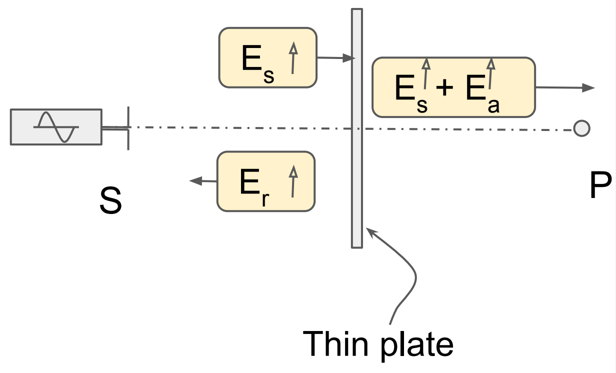


Fig 4-11

Opaque plate

For an opaque plate, the field at point P is zero

$$E_P = E_s + E_a = 0 \rightarrow E_a = -E_s$$ Eq 4-12

For an opaque plate with an aperture, the field at point P can be written as

$$E_P = E_s + E_w,\ E_w \text{ is the field from the walls}$$ Eq 4-13

If we plug the aperture with the same opaque material, then the total field at P is zero again.

$$E_s + E_w' + E_{pl} = 0$$ Eq 4-14

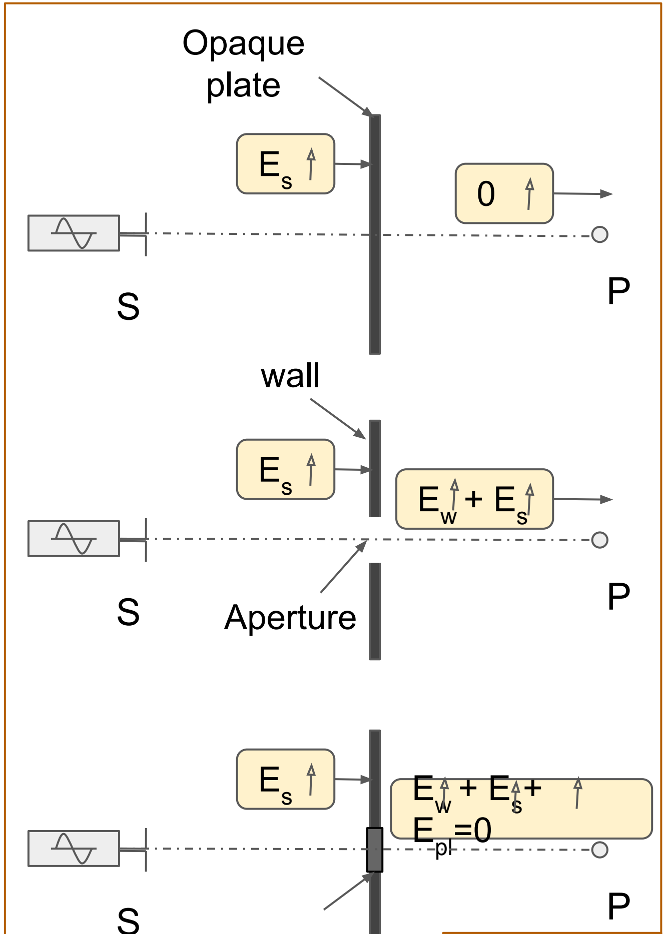


Fig 4-12

Diffraction by an aperture

In Eq 4-14 the field emitted due to the wall charges with the plug present is marked with a prime to indicate a difference from the case of aperture. When an aperture present, the field emitted by the wall is slightly different due to charges at the edge of the aperture. Substituting Eq 4-14 in 4-13

$$E_p = E_w - E_w' - E_{pl} \quad \text{Eq 4-15}$$

If one neglects such effect and let $E_w = E_w'$, the the total field at point is

$$E_p = -E_{pl} \quad \text{Eq 4-16}$$

This result states that the field at a far observer due to the presence of an aperture is equivalent to the field produced by filling the aperture with charges which oscillate by a force exerted by the incident electric field.

Diffraction by two apertures

Now, let us place two apertures each has a width d and they are spaced by a distance L (center to center). If the top aperture is assume to go through the origin, then the field component A can be written as

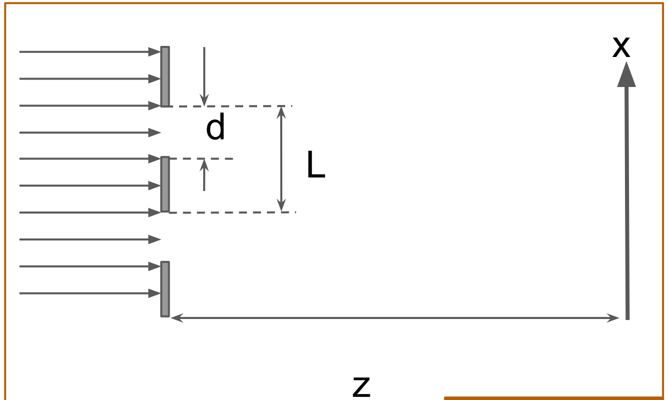


Fig 4-13

$$A = \sum_{n=0}^{N} e^{in\Delta\phi} + \sum_{n=0}^{N} e^{i\frac{2\pi}{\lambda}(n\delta + L)\sin\theta} = \sum_{n=0}^{N} e^{in\Delta\phi} + \sum_{n=0}^{N} e^{i\left(n\Delta\phi + \frac{2\pi}{\lambda}L\sin\theta\right)}$$

Eq 4-17

$$A = \sum_{n=0}^{N} e^{in\Delta\phi}\left(1 + e^{i\psi}\right), \quad where \quad \psi = \frac{2\pi}{\lambda}L\sin\theta$$

Eq 4-18

Aperture size effect

The intensity can be calculated using Eq 4-?

$$I \propto \left(\frac{\sin^2(\phi/2)}{(\phi/2)^2} \right) \cos^2(\psi/2)$$

Eq 4-19

Increasing the aperture size, d, the first term in Eq 4-19 Becomes narrower, which works as an envelope of the fringes. Narrower envelope causes more diffraction orders to be suppressed.

For narrow aperture, the envelope term is wide and the results become similar to two dipoles case.

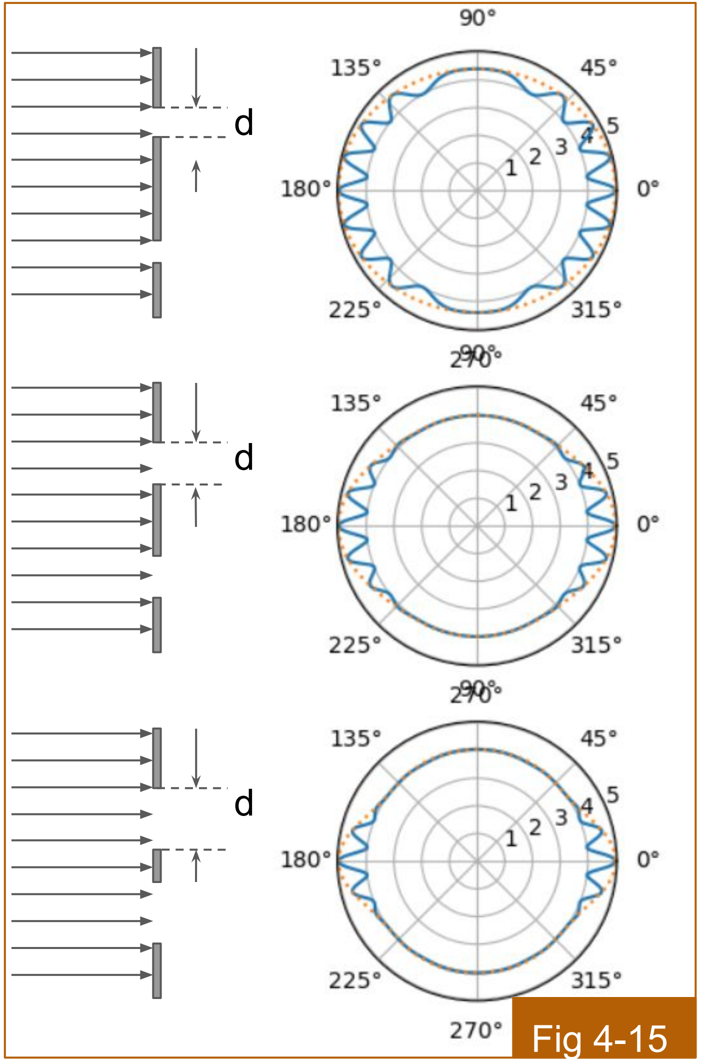


Fig 4-15

Increasing the number of apertures

What happens if we increase the number of apertures keeping fix spacing, L, between any two neighbouring apertures?

Eq 4-17 will need to include more terms for each aperture. If we assume a total of M apertures, then

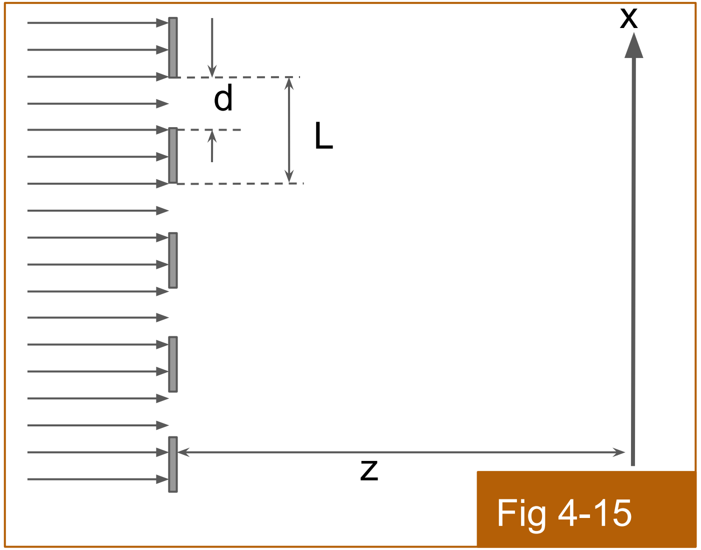


Fig 4-15

$$A = \sum_{n=0}^{N} e^{in\Delta\phi} + \sum_{n=0}^{N} e^{in\Delta\phi + i\psi} + \ldots + \sum_{n=0}^{N} e^{in\Delta\phi + i(M-1)\psi} + \sum_{n=0}^{N} e^{in\Delta\phi + iM\psi}$$

Eq 4-20

Diffraction by periodical apertures

Eq 4-20 can be written as

$$A=\sum_{m=0}^{M}\sum_{n=0}^{N}e^{i(n\Delta\phi+m\psi)}$$

Eq 4-21

$$A=\sum_{m=0}^{M}\left(N\frac{\sin\phi}{\phi}\right)e^{im\psi}=\left(N\frac{\sin\phi}{\phi}\right)\sum_{m=0}^{M}e^{im\psi}$$

Eq 4-22

$$A=\left(N\frac{\sin(\phi/2)}{(\phi/2)}\right)\left(\frac{\sin(M\psi/2)}{\sin(\psi/2)}\right)$$

Eq 4-23

Effect of multiple apertures

Notice that in Eq 4-23, we did not approximate sin(ψ/2) to be ψ/2. This is the due to the fact that ψ is not small as was the case of Δϕ.

Having multiple apertures with the same spacing (period) causes narrow diffraction orders.

This structure is commonly referred to as diffraction grating.

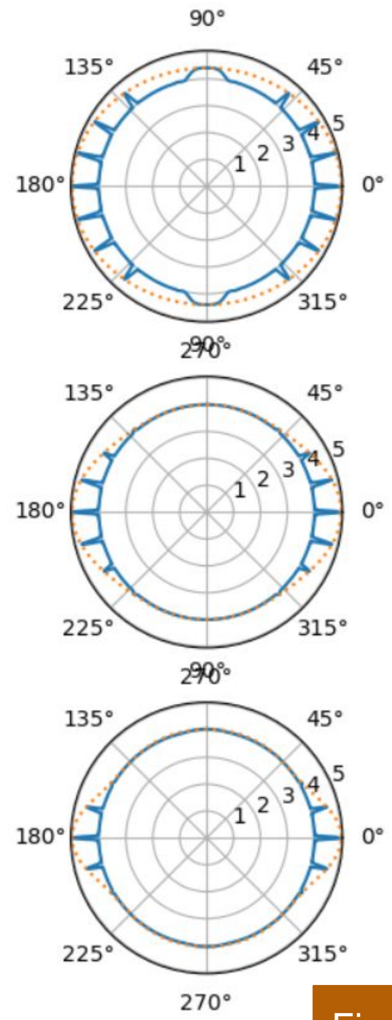


Fig 4-16

Two apertures vs. multiple apertures

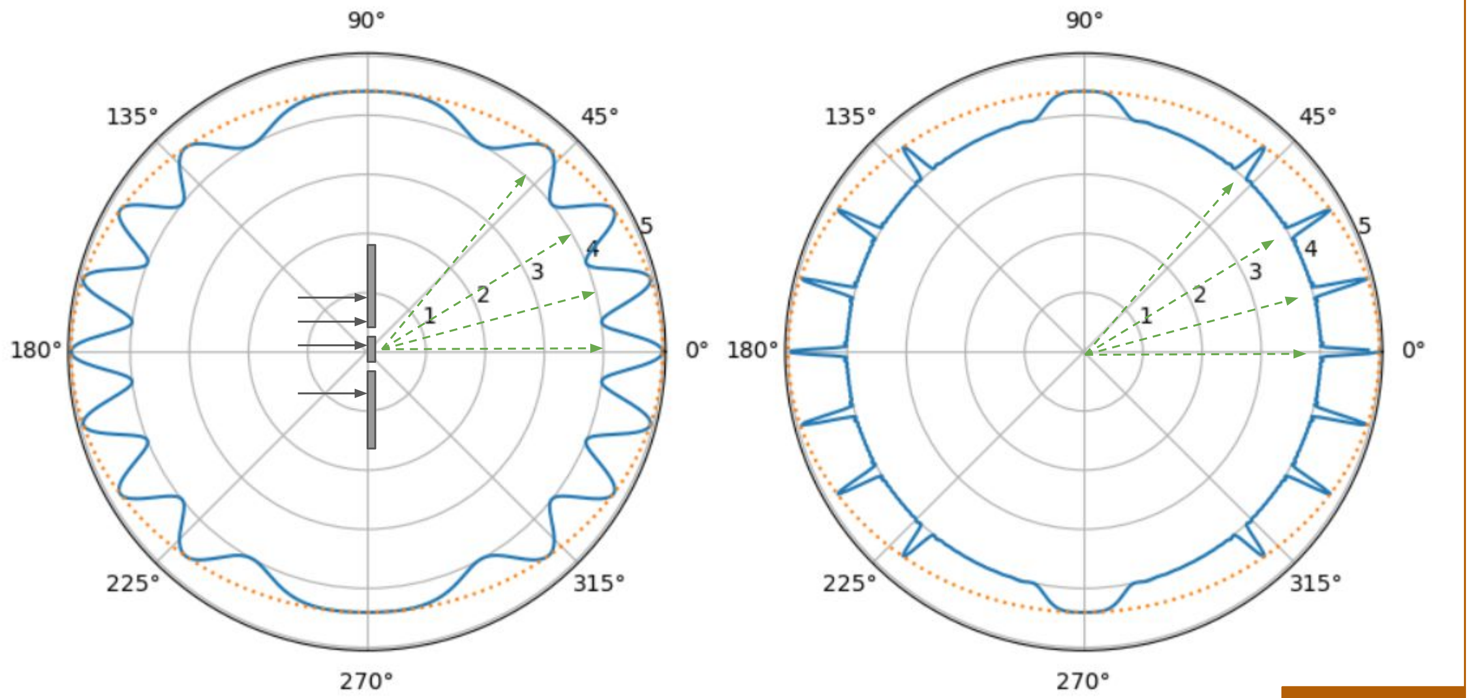


Fig 4-16