Part 5: Refractive index

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Light matter interaction model

In the previous part, we illustrated that diffraction by an aperture is equivalent to filling it with oscillating dipoles and observing the field at a point far from it. In doing so, a crude assumption was made that the field emitted by each dipole is not affected by the surrounded charges. This works for a medium of low charge density such as gas where the distance between particles is large enough to ignore their interaction. Surprisingly, this approximation can still be able to describe light interaction with denser media such dielectrics. For now, let us examine the interaction of light incident on a low density medium and try to find the fields at a point P far behind the medium.

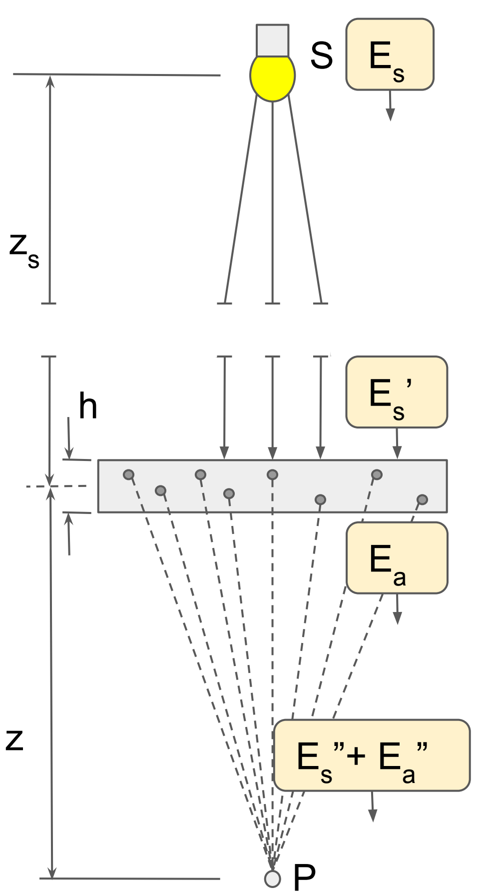

Fig 5-1

Field components

in figure 5-1, the observer at point P sees a total field as a sum of the effects due to all charges including the source. The observed field can be divided into two parts: the one due to source (E_s) and one due to all other charges (E_a).

At the medium, E_s reaches all charges with a retardation Z_s/c neglecting the medium width $h << Z_s$.This field exerts a force on each charge forming oscillating dipoles. The observer at P sees the sum of all the fields generated by each dipole. To find an expression for E_a one can follow the following assumptions:

1- E_s causes the charges in the medium to oscillate.
2- The oscillating charges emit a total field of E_a.
3- Only the medium charges contribute to E_a.

Total field at the observer

At point P, the source arrives with a total retardation of $(Z+Z_s)/c$.

$$E_s' = E_s(t - \frac{Z_s}{c}) \text{ and } E_s'' = E_s'(t - \frac{Z}{c}) \rightarrow E_s'' = E_s(t - \frac{Z+Z_s}{c}) \quad \text{Eq 5-1}$$

$$E_P = E_s'' + E_a \quad \text{Eq 5-2}$$

Now it is time find E_a. The field is a result from all the charges in the medium volume. Though the problem seems to be complicated, one can simplify the calculations if we assume a uniform distribution of charges in the volume.

For *N* charges per unit volume, the surface charge density is η = *N h* where h is the medium thickness. This approximation can simplify the analysis to first calculating the total field due to a sheet of charges then include the volume effect.

Sheet of infinite dipoles

Consider a plane sheet that is filled uniformly with exact dipoles with η charges per unit area. Without loss of generality one can choose the coordinate system such that the point P falls on the z axis.

The field at P generate by an infinitesimal area dA = $\rho d\rho d\varphi$ is assumed to be a result of ηdA charges.

$$dE_a = \frac{-\eta q a_o}{4\pi\epsilon_o c^2 r} e^{i(\omega t - \omega r/c)} \rho d\rho d\varphi$$

Eq 5-3

The total field can be calculated as the integration over the entire sheet area.

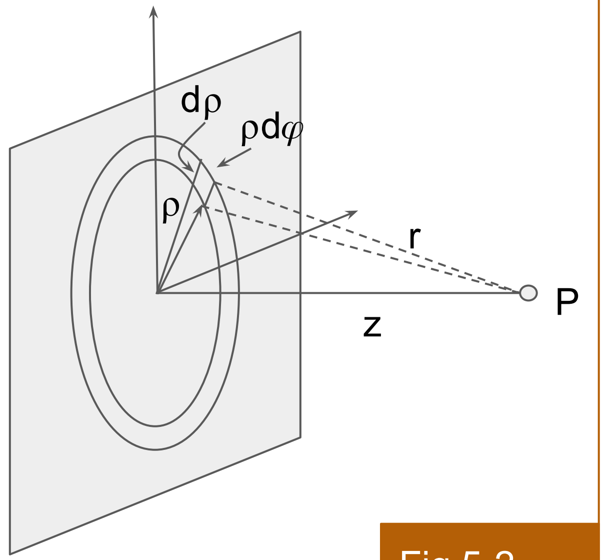

Fig 5-2

Total field due to a sheet of dipoles

$$E_a = \int_{\varphi=0}^{2\pi} \int_{0}^{\infty} \frac{-\eta q a_o}{4\pi \epsilon_o c^2 r} e^{i(\omega t - \omega r/c)} \rho d\rho d\varphi$$ Eq 5-4

Knowing $r^2 = \rho^2 + z^2 \rightarrow rdr = \rho d\rho$ then

$$E_a = \frac{-\eta q a_o}{2\epsilon_o c^2} \int_{z}^{\infty} e^{i(\omega t - \omega r/c)} dr$$ Eq 5-5

$$E_a = \frac{\eta q a_o}{2\epsilon_o i \omega c} [e^{i\infty} - e^{i(\omega t - \omega z/c)}]$$ Eq 5-6

For now let us set the term $e^{-i\infty}$ to zero. We will explain later why this is a valid physical assumption. For now , using the relation $a_o = -\omega^2 x_o$ Eq 5-6 becomes

$$E_a = -i \frac{\eta q \omega x_o}{2\epsilon_o c} e^{i(\omega t - \omega z/c)}$$ Eq 5-7

Total field due to charges in the medium volume

Following the approximation η = *N h*, the field due to all medium charges is

$$E_a = -i \frac{N h q \omega x_o}{2 \epsilon_o c} e^{i(\omega t - \omega z/c)}$$ Eq 5-8

One needs to keep in mind that in this approximation the thickness effect is neglected as the medium is assumed to be thin or h<<Z.

In Eq 5-8, the term $i \omega x_o e^{i(\omega t - \omega z/c)}$ is the speed of a charge linearly oscillating as the source with a z/c time retardation. The real part of this term is the time derivative of Eq 2-1.

$$E_a = \frac{Nhq}{2\epsilon_o c} v_p$$ Eq 5-9

Radiated power

This is the same result we obtained earlier in Eq 2-33 when multiplying the numerator and denominator by Nh. Equating Eq 5-9 and Eq 2-33, one can find a solution for the proportionality constant α.

$$\frac{Nhq}{2\epsilon_o c} v_p = \frac{Nhq}{2Nh\alpha} \rightarrow Nh\alpha = \eta\alpha = \epsilon_o c$$

Eq 5-10

Using the expressing in Eq 5-10, the intensity in Eq 2-41 can now be written as

$$\langle I \rangle = \epsilon_o c \langle E^2 \rangle$$

Eq 5-11

Here, the suffix s is removed as the expression is not restricted only to the source radiation.

Harmonic oscillator

In the medium, the charges are the electrons which are bound to their atoms. These charges feel the applied field and oscillate accordingly. However they are fastened elestically to their atom similar to a mass attached a spring (with a spring constant k). If the electron was shifted by a distance x_o and with no force applied during the motion, its location at any point of time can be approximated as

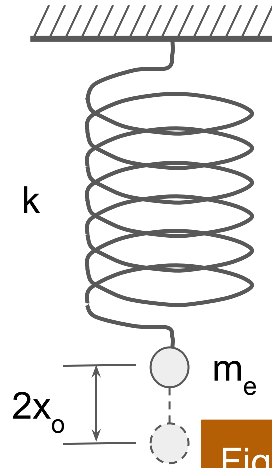

Fig 5-3

$$-kx = m_e \frac{d^2 x}{dt^2} \rightarrow x(t) = x_o e^{i\omega_o t}, \text{ where } \omega_o = \sqrt{\frac{k}{m_e}}$$ Eq 5-12

In Eq 5-9, *-kx* is the restoring force by the atom.

Motion under an applied force

When a force is applied on the electron by the incident source electric field, the electron is forced to linearly oscillate with the field. The equation of motion is now

$$qE_s' - kx = m_e \frac{d^2x}{dt^2} \quad \text{, where } x(t) = x_o e^{i(\omega t)} \qquad \text{Eq 5-13}$$

Substituting the expression of x into the left side of Eq 5-13

$$\frac{qE_s'}{m_e} - \omega_o^2 x_o e^{i\omega t} = -\omega^2 x_o e^{i\omega t} \rightarrow x_o = \frac{-qE_s'}{m_e(\omega^2 - \omega_o^2)} e^{-i\omega t} \qquad \text{Eq 5-14}$$

Substituting the result in Eq 5-14 into Eq 5-8

$$E_a = i \frac{Nhq^2 \omega E_s'}{2\epsilon_o c m_e(\omega^2 - \omega_o^2)} e^{-i\omega z/c} \qquad \text{Eq 5-15}$$

Source field

The total field from Eq 5-2 is now expressed as

$$E_P = E_s'' + E_a = E_s'' + i\frac{N h q^2 \omega E_s'}{2\epsilon_o c m_e(\omega^2 - \omega_o^2)} e^{-i\omega z/c}$$ Eq 5-16

Now it is time to look into finding expressions for the source field Es' and Es''. One can for simplicity assume the field coming from a sheet of uniform dipoles as we did with the medium. However here the dipoles are assumed to oscillate (due to electrical force or other source of excitation.) Hence, the field at any far point from the source can be written as (the suffix *s* indicates that is the source property)

$$E_s(z) = E_o e^{i(\omega t - \omega z/c)}$$, where Eq 5-17

$$E_o = \frac{-i\eta_s q x_{o,s}\omega}{2\epsilon_o c}$$ Eq 5-18

Source propagation

The expression in Eq 5-17 shows a very interesting result. It says that if we freeze time and shift the location of the observer (by say Δz) the measured field only experiences a phase shift of $\omega \Delta z/c$. One might question what about the previous results showing the amplitude reduction of $1/z$. If the distance z is very large and Δz is much smaller than z then one can easily assumes that $z+\Delta z \cong z$. Hence the distance Z_s in Fig 5-1 has to be much larger than any other dimensions in the model. Using Eq 5-17, one can re-write Eq 5-16 as

$$E_P=\left(1+i\frac{N h q^2 \omega}{2\epsilon_o c\, m_e(\omega^2-\omega_o^2)}\right)E_o e^{i\left(\omega t-\frac{\omega}{c}(Z_s+Z)\right)} \qquad \text{Eq 5-19}$$

Phase difference

Let us now assume that the frequency of the source oscillation, ω, is set to be much larger than ω_o. The imaginary part in the baranthes in Eq 5-19 becomes much smaller than 1. One then can apply the following assumption $e^{i\alpha} \approx 1 + i\alpha$. We can write Eq 5-16 as

$$E_P \approx e^{i\left(\frac{N h q^2 \omega}{2 \epsilon_o c m_e (\omega^2 - \omega_o^2)}\right)} E_o e^{i\left(\omega t - \frac{\omega}{c}(Z_s + Z)\right)} = E_o e^{i\left(\omega t - \frac{\omega}{c}\left(Z_s + Z - h\left[\frac{N q^2}{2 \epsilon_o m_e (\omega^2 - \omega_o^2)}\right]\right)\right)}$$ Eq 5-20

If the medium did not exist, the observer at point P would measure

$$E_p'(Z_s + Z) \approx E_o e^{i\left(\omega t - \frac{\omega}{c}(Z_s + Z)\right)}$$ Eq 5-21

That means that the presence of the medium causes a phase shift of

$$\Delta \phi = -\frac{\omega}{c} h \left[\frac{N q^2}{2 \epsilon_o m_e (\omega^2 - \omega_o^2)}\right]$$ Eq 5-22

Refractive index

If light would have propagated in the same thickness d without the medium the phase gained would have been $\phi_{air}=\frac{\omega}{c}h$. Hence, the phase accumulated by the light propagating through the medium is $\phi_{medium}=\phi_{air}+\Delta\phi$

$$\phi_{medium}=\frac{\omega}{c}h[1+\frac{N q^2}{2\epsilon_o m_e(\omega_o^2-\omega^2)}]$$ Eq 5-23

One can now say that the speed of light inside the medium is reduced by a factor n, referred to as the refractive index.

$$\phi_{medium}=\frac{\omega}{c_{medium}}h \text{ , where } c_{medium}=\frac{c}{n}$$ Eq 5-24

$$n=1+\frac{N q^2}{2\epsilon_o m_e(\omega_o^2-\omega^2)}$$ Eq 5-25

Plasma frequency

Interestingly enough the term $N q^2 / \epsilon_o m_e$, which is present in Eq 5-25, is the square of a quantity known as the plasma frequency, ω_p. This is the frequency of oscillation of electrons about their equilibrium state, an instability in the dielectric function in a conducting medium such as plasma or metal, when no external field is applied, $\omega_p = \sqrt{N q^2 / \epsilon_o m_e}$.

$$n = 1 + \frac{\omega_p^2}{2(\omega_o^2 - \omega^2)} \qquad \text{Eq 5-26}$$

Looking at Eq 5-26, if the light frequency approaches ω_o, the refractive index would increase rather rapidly to infinity. This is the result of ignoring the damping factor in the oscillation due to energy emitted by the moving electrons.

Damping oscillation

If we incorporate a damping factor γ, the equation of electron motion becomes

$$qE_s' - kx - m_e \gamma \frac{dx}{dt} = m_e \frac{d^2 x}{dt^2} \quad \text{, where } x(t) = x_o e^{i\omega t} \rightarrow$$

Eq 5-27

$$x_o = \frac{qE_s'}{(\omega_o^2 - \omega^2) + i\gamma\omega} e^{-i\omega t}$$

Eq 5-28

The refractive index is then written as

$$n = 1 + \frac{1}{2} \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + i\gamma\omega}$$

Eq 5-29

Complex refractive index

We can write the refractive index in terms of real and imaginary parts

$$\tilde{n}=1+\frac{1}{2}\omega_p^2\frac{(\omega_o^2-\omega^2)-i\gamma\omega}{(\omega_o^2-\omega^2)^2+(\gamma\omega)^2}=n+i\kappa$$ Eq 5-30

Note that the tilde sign is now added to indicate complex refractive index in general keeping the symbol n for the real part.

$$n=1+\frac{\omega_p^2(\omega_o^2-\omega^2)}{2[(\omega_o^2-\omega^2)^2+(\gamma\omega)^2]}$$ Eq 5-31

$$\kappa=\frac{\omega_p^2(\omega\gamma)}{2[(\omega_o^2-\omega^2)^2+(\gamma\omega)^2]}$$ Eq 5-32

Near resonance

When the frequency of the incident light approaches the natural frequency, ω_o, the real part n increases then decreases to values less than unity and it increases back again.

The imaginary part however peaks at ω_o and rapidly decays away from it.

The condition at which $\omega = \omega_o$ is referred to as <u>resonance</u>.

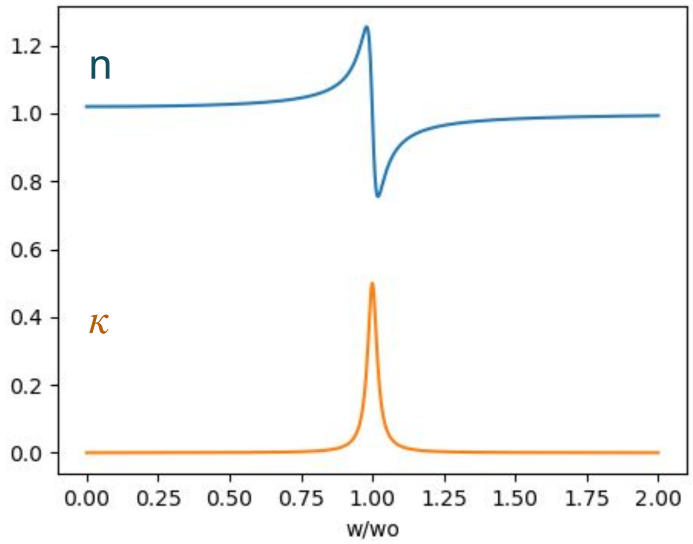

Fig 5-4

Refractive index limits

Let us write Eq 5-31 as follows

$$n=1+\frac{1}{2}\frac{\omega_p^2}{\omega_o^2-\omega^2+\frac{\gamma^2}{\left(\omega_o^2-\omega^2\right)}\omega^2}$$ Eq 5-33

Let us examine the limites

1- For frequencies much lower than natural frequency, $\omega \ll \omega_o$, Eq 5-33 can be simplified as

$$n\approx 1+\frac{1}{2}\frac{\omega_p^2}{\omega_o^2-\omega^2}\approx 1+\frac{1}{2}\frac{\omega_p^2}{\omega_o^2}$$ Eq 5-34

2- For frequencies much larger than natural frequency, $\omega \gg \omega_o$, Eq 5-33 can be simplified as

$$n\approx 1-\frac{1}{2}\frac{\omega_p^2}{\omega^2-\omega_o^2+\gamma^2}\approx 1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}$$ Eq 5-35

Multiple resonances

If the molecules forming the medium have multiple natural frequencies then Eq 5-29 needs to be modified to accommodate for these resonances. If there are N_k electronics per unit volume have resonance $\omega_{o,k}$ then the complex refractive index can be written as

$$\tilde{n} = 1 + \frac{1}{2}\sum_k \frac{\omega_{p,k}^2}{\omega_{o,k}^2 - \omega^2 + i\,\gamma_k\,\omega}$$

Eq 5-36

where

$$\omega_{p,k}^2 = N_k \frac{q^2}{\epsilon_o m_e}$$

Eq 5-37

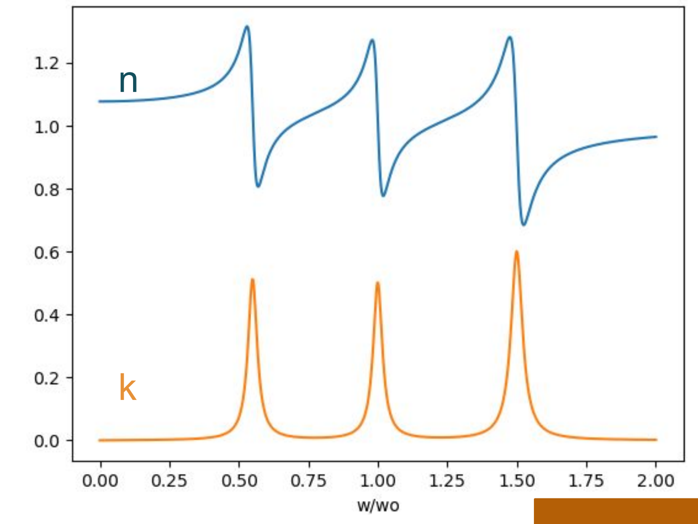

Fig 5-5